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## Real and Harmonic Analysis

Exercise Sheet 2 | 15 Oct, 2014 | Turn in by: 8:00 22 Oct, 2014

### Problem 4 (10 points)

1. *3 points* Suppose that $f \in L^1(\mathbb{T})$ and that $S_N f = \sum_{n=-N}^{N} \widehat{f}(n) e^{i2\pi nx}$ is such that for some $p \in [1;\infty]$ we have that $S_N f \to g$ in $L^p(\mathbb{T})$ for some $g \in L^p(\mathbb{T})$. Prove that $f = g$ and if $p = \infty$ then $f \in C(\mathbb{T})$.

2. *2 points* Is the Fourier transform defined on Radon measures $\mathcal{M}_{Rad}(\mathbb{T})$ as
$$\widehat{\mu}(n) \overset{\text{def}}{=} \int_{\mathbb{T}} e^{-i2\pi nx} \mathrm{d}\mu(x)$$
injective?

3. *3 points* Is the embedding $H^r(\mathbb{T}) \hookrightarrow H^s(\mathbb{T})$ for $r > s$ a compact map? Is the embedding map $C^r(\mathbb{T}) \hookrightarrow H^s(\mathbb{T})$ compact?

4. *2 points* Show that for any $\omega : \mathbb{N} \cup \{0\} \to \mathbb{R}^+$ with $\lim_{n\to\infty} \omega(n) = 0$ there exists a measurable set $E \subset \mathbb{T}$ s.t. $\limsup_{n\to\infty} \frac{\left|\widehat{\mathbb{1}}_E(n)\right|}{\omega(n)} = \infty$.

### Problem 5 (10 point)

1. *5 points* Let
$$M_0 f(x) \overset{\text{def}}{=} \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \mathrm{d}y$$
be the centered maximal function. Prove that $C^{-1} M f(x) \leq M f(x) \leq C M_0 f(x)$ for some $C > 1$. Let
$$M_b f(x) \overset{\text{def}}{=} \sup_{r>0} \sup_{|x'-x|<br} \frac{1}{|B(x,r)|} \int_{B(x',r)} |f(y)| \mathrm{d}y$$
with $b > 0$ be the so called non-tangential maximal function. Is it true that $M_b f(x) \leq C_b M f(x)$ and $M f(x) \leq C_b M_b f(x)$ for some $C_{b,n} > 1$? In that case, how does $C_b$ depend on $b$ and the dimensions $n$ of $\mathbb{R}^n$? Let
$$M_N f(x) \overset{\text{def}}{=} \sup_{r>0} \left(1 + \frac{|x-x'|}{r}\right)^{-N} \frac{1}{|B(x,r)|} \int_{B(x',r)} |f(y)| \mathrm{d}y$$

with $b > 0$ and some large $N > 0$ be the so called strongly non-centered maximal function. Which of the following inequalities hold: $M_N f(x) \le C_N M f(x)$ and $M f(x) \le C_N M_N f(x)$?

2. *5 points* Prove the point-wise bound $K_n * f \le C\, Mf$. Let $\Phi_n$ be an approximate identity and let $f \in L^p(\mathbb{R})$ for $p \in [1, \infty)$. Does $\Phi_n * f \to f$ point-wise a.e.?

**Problem 6 (10 points)**

We want to prove the Hardy inequality:

$$(1) \qquad \left\| x^{s-1} f(x) \right\|_{L^p(\mathbb{R}^+)} \le \frac{1}{1 - s - 1/p} \left\| x^s f'(x) \right\|_{L^p(\mathbb{R}^+)} \qquad f \in C^1\left([0; \infty)\right),\ f(0) = 0$$

$$(2) \qquad \left\| x^{s-1} \int_0^x g(s) \mathrm{d}s \right\|_{L^p(\mathbb{R}^+)} \le \frac{1}{1 - s - 1/p} \left\| x^s g(x) \right\|_{L^p(\mathbb{R}^+)} \quad g \in L^1(\mathbb{R}^+)$$

1. *5 points* Show that the two above inequalities are equivalent. Use Fubini to determine for which $s \in \mathbb{R}$ the above inequalities hold for $p = 1$. By setting $f = h^p$ in the first inequality determine for each $p \in [1, \infty]$ for which range $s \in \mathbb{R}$ the above inequalities hold.

2. *5 points* We now want to prove the same result using interpolation. Set $T_{z,s} g(x) = x^{s-1+z} \int_0^x y^{-s-z} g(y) \mathrm{d}y$ for $z \in \left\{ z \in \mathbb{C} |\ \Re z \in [0, 1] \right\}$. Determine for which $s \in \mathbb{R}$ the family of operators $T_{z,s}$ are abounded on $L^1(\mathbb{R}^+)$ for $\Re z = 0$ and on $L^\infty(\mathbb{R}^+)$ for $\Re z = 1$. Does this family satisfy the hypothesis of Riesz-Thorin Theorem on $\Re(z) \in [0; 1]$. Apply Riesz-Thorin to the family $T_{z,s}$ to prove Hardy inequality.

**All answers should be fully justified. The solution to a given point may depend on the solutions of the previous points: if you didn't manage to prove a required statement you can still use it in the subsequent points. Working in small groups is encouraged (no more than 4 people/group). You may hand in only one answer sheet per group. Put all the names on the sheet. During the exercise sessions students must know how to solve all the exercises done by their group, otherwise the assignment is void.**