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Real and Harmonic Analysis

Exercise Sheet 1 | 08 Oct, 2014 | Turn in by: 8:00 15 Oct, 2014

Problem 1 (10 points)

Definition 1 (Algebra)**.** *A vector space $\mathbb{X}$ endowed with a bilinear map $(x, y) \mapsto xy \in \mathbb{X}$ is called an* algebra. *If $\mathbb{X}$ is a Banach space we require that bilinear map satisfy $\|xy\|_{\mathbb{X}} \leq \|x\|_{\mathbb{X}} \|y\|_{\mathbb{X}}$.*
A (Banach) algebra $\mathbb{X}$ is associative *if $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{X}$. It is* commutative *if $xy = yx$ for all $x, y \in \mathbb{X}$. An element $u \in \mathbb{X}$ is a unit if $ux = xu = x$ for all $x \in \mathbb{X}$. An algebra with a unit is called* unital.
Notice that an associative unital algebra is a ring.

1. *2 points* Recall the defintion of the convolution of two L^1 functions on $\mathbb{R}$. The Gaussian functions on $\mathbb{R}$ are $N(c, \sigma) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-c)^2}{2\sigma^2}}$ with $c \in \mathbb{R}$ and $\sigma > 0$. Compute the convolution of two Gaussian functions: $N(c_1, \sigma_1) * N(c_2, \sigma_2)$. Is it a Gaussian?

   Hint: $\|N(x, \sigma)\|_{L^1} = 1$.

2. *2 points* Recall the definition of convolution of two L^1 functions on $\mathbb{T}$, $\mathbb{R}$ and $\mathbb{Z}$. Show that $(L^1, *)$ is an associative commutative algebra. Show that L^∞ functions on $\mathbb{T}, \mathbb{R}, \mathbb{Z}$ endowed with multiplication $\cdot$ form an algebra. Is the Fourier transform $\mathcal{F} : f \mapsto \widehat{f}$ a homeomorphism of Banach algebras from $(L^1(\mathbb{T}), *)$ to $(l^\infty(\mathbb{Z}), \cdot)$?

3. *3 points* Prove the Riemann-Lebesgue Lemma:
   Lemma 2 (Riemann-Lebesgue)**.** *Let $f \in L^1(\mathbb{T})$ then $\lim_{n \to \pm\infty} \widehat{f}(n) = 0$.*

   Hint: Approximate $f \in L^1(\mathbb{T})$ in $\|\cdot\|_{L^1(\mathbb{T})}$ for example using the Fejér kernels K_n.

4. *3 points* An idempotent element of an algebra $\mathbb{X}$ is an element $x \in \mathbb{X}$ such that $x \cdot x = x$. Find the idempotent elements of $(L^1(\mathbb{T}), *)$.

Problem 2 (10 points)

Definition 3 (Finite Radon measures)**.** *The dual of $C_b(\mathbb{R})$, the continuous bounded functions on $\mathbb{R}$ endowed with the norm $\|f\|_{sup} = \sup_{x \in \mathbb{R}} |f(x)|$ are* finite Radon measures $\mathcal{M}_{Rad}(\mathbb{R})$. *The natural norm on $\mathcal{M}_{Rad}(\mathbb{R})$ is given by duality: $\|\mu\|_{\mathcal{M}_{Rad}} = \sup_{\|f\|_{sup} < 1} \left|\int f \mathrm{d}\mu\right|$. The definition of $\mathcal{M}_{Rad}(\mathbb{T})$ is similar.*

We now want to extend the notion of convolution to $\mathcal{M}_{Rad}(\mathbb{T})$ and $\mathcal{M}_{Rad}(\mathbb{R})$.

1. *3 points* Prove that given $\mu, \nu \in \mathcal{M}_{Rad}(\mathbb{T})$, the map $f \mapsto \iint_{\mathbb{T}\times\mathbb{T}} f(x+y)\mathrm{d}\mu(x)\mathrm{d}\nu(y)$ is well defined. Deduce that it uniquely defines a measure $\lambda \in \mathcal{M}_{Rad}(\mathbb{T})$ and we define $\mu * \nu \stackrel{\text{def}}{=} \lambda$.

2. *2 points* Is $\left(\mathcal{M}_{Rad}(\mathbb{T}), *\right)$ a topological algebra? $L^1(\mathbb{T})$ is cannonically embedded in $\mathcal{M}_{Rad}(\mathbb{T})$ via $f \hookrightarrow f\mathrm{d}\Lambda$ where Λ is the Lebesgue measure on $\mathbb{T}$. Is this embedding an embedding of topological algebras? Is $\left(\mathcal{M}_{Rad}(\mathbb{T}), *\right)$ associative? Is it commutative? Does $\left(\mathcal{M}_{Rad}(\mathbb{T}), *\right)$ have a unit? If it does, exhibit it explicitly.

   Notice that the above procedure can also be used to define $\left(\mathcal{M}_{Rad}(\mathbb{R}), *\right)$.

3. *3 points* Show that the convolution map $(f, g) \mapsto f * g$ is continuous from $L^p(\mathbb{T}) \times L^{p'}(\mathbb{T}) \to C_b(\mathbb{T})$ where the latter space is endowed, as usual, with the $\sup$ norm.

   Hint: Use the continuity in $L^p(\mathbb{T})$ of the translation map $\tau : z \mapsto f(\cdot - z)$

4. *2 points* Do the algebras $\left(L^1(\mathbb{T}), *\right)$ and $\left(L^1(\mathbb{R}), *\right)$ have units?

Problem 3 (10 points)

We observed that if a function is in some Hölder space i.e. $f \in C^\alpha(\mathbb{R})$ then $\sum_{-N}^{N} \widehat{f}(k)e^{2\pi ikx} \to f$ uniformly. We also know that if $f \in L^1$ we have $\widehat{f} \in l^\infty(\mathbb{Z})$. We would like to state a finer, more quantitative result.

1. *5 points* Given $\alpha \in (0, 1)$ let $f \in C^\alpha(\mathbb{T})$. Prove that there exists $\beta > 0$ such that $\sup_{n\in\mathbb{Z}} \left|\widehat{f}(n)\left(1 + |n|\right)^\beta\right| < \infty$. Find all such β. Express the bounds on $\left|\widehat{f}(n)\left(1 + |n|\right)^\beta\right|$ in terms of the α-Hölder semi-norm $\|\cdot\|_{\mathring{C}^\alpha}$ and the $\|\cdot\|_{sup}$ norm of f.

2. *5 points* Now conversely, suppose that $f \in C^0(\mathbb{T})$ and there is a constant $C > 0$ such that $\left|\widehat{f}(n)\left(1 + |n|\right)^\beta\right| < C$ for all $n \in \mathbb{Z}$. Is it true that there exists some $\alpha > 0$ so that necessarily $f \in C^\alpha(\mathbb{T})$? Is it true that necessarily $f \in C^\beta(\mathbb{T})$? If not provide a counter-example. Determine all $\alpha > 0$ for which necessarily $f \in C^\alpha(\mathbb{T})$.

All answers should be fully justified. The solution to a given point may depend on the solutions of the previous points: if you didn't manage to prove a required statement you can still use it in the subsequent points. Working in small groups is encouraged (no more than 4 people/group). You may hand in only one answer sheet per group. Put all the names on the sheet. During the exercise sessions students must know how to solve all the exercises done by their group, otherwise the assignment is void.