HARMONIC ANALYSIS SUMMER 2020, PART B

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8. Lecture: Martingale differences and paraproducts

A product of martingales $F(I)G(I)$ is typically not a martingale, as it does not satisfy the martingale identity. In particular, if the limit

$$\lim_{k\to-\infty} \sum_{k\in\mathcal{D}_k} |I|F(I)G(I)f(c(I)) \tag{1}$$

exists and defines a Radon measure, the martingale extension of that measure is not $F(I)G(I)$. If F and G are martingale extensions of integration against functions f, g in $\mathcal{S}^\Delta$, then for sufficiently small I we have that $F(I) = F(I_l) = F(I_r)$ and similarly for G, so that the product $F(I)G(I)$ satisfies the martingale identity and equals the martingale extension H of integration against $h = fg$ at least for sufficiently small I.
To better understand products of Radon measures and related issues, we develop the theory of martingale differences and paraproducts.
We begin with the modulus square of a martingale.

Theorem 36. *Let $F : \mathcal{D} \to \mathbb{C}$ satisfy the martingale identity. If*

$$\sup_k \sum_{k\in\mathcal{D}_k} |I|F(I)\overline{F(I)} < \infty, \tag{2}$$

Then the limit

$$m(f) := \lim_{k\to-\infty} \sum_{k\in\mathcal{D}_k} F(I)\overline{F(I)}f(c(I)) \tag{3}$$

exists for all $f \in \mathcal{S}^\Delta$ and defines a nonnegative Radon measure.

Proof. Define the martingale difference $\Delta F(I)$ by

$$|I|\Delta F(I) = |I_l|F(I_l) - |I_r|F(I_r).$$

Then we have the parallelogram law

$$|I|F(I)\overline{F(I)} + |I|\Delta F(I)\overline{\Delta F(I)} = |I_l|F(I_l)\overline{F(I_l)} + |I_r|F(I_r)\overline{F(I_r)}.$$

Summing over $I \in \mathcal{D}_k$ we obtain

$$\sum_{I\in\mathcal{D}_k} |I||F(I)|^2 + \sum_{I\in\mathcal{D}_k} |I||\Delta F(I)|^2 = \sum_{I\in\mathcal{D}_{k-1}} |I||F(I)|^2.$$

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Telescoping in the scale gives for $k < k'$

$$\sum_{I\in\mathcal{D}_{k'}} |I||F(I)|^2 + \sum_{k<j\le k'} \sum_{I\in\mathcal{D}_j} |I||\Delta F(I)|^2 = \sum_{I\in\mathcal{D}_k} |I||F(I)|^2.$$

In particular, the sequence $\sum_{I\in\mathcal{D}_k} |I||F(I)|^2$ increases as $k \to -\infty$. It also remains bounded by the assumption in the theorem, and we obtain

$$|F([0,1))|^2 + \sum_{I\in\mathcal{D}} |I||\Delta F(I)|^2 < \infty.$$

For k smaller than the scale of f, we have

$$|\sum_{I\in\mathcal{D}_k} |I||F(I)|^2 f(c(I) - \sum_{I\in\mathcal{D}_{k-1}} |I||F(I)|^2 f(c(I)| \le \|f\|_\infty \sum_{I\in\mathcal{D}_k} |I||\Delta F(I)|^2$$

It follows that the sequence inside the limit in (3) is Cauchy and therefore the limit exists. The limit defines a linear functional on $\mathcal{S}^\Delta$ that is bounded with the constant (2). □

Note the importance in this proof of telescoping the limits into sums over the collection of $\mathcal{D}$ to use the theory of absolutely summable series. The following theorem provides the algebraic procedure for more general products.

Theorem 37 (Telescoping identity). *Let $n \ge 2$. Let F_j for $1 \le j \le n$ be martingales. Let $N = \{1, \dots, n\}$ and let $\mathcal{E}$ be the set of nonempty subsets of N with even cardinality. Then*

$$-\prod_{j\in N} F_j([0,1)) + \sum_{I\in\mathcal{D}_k} |I| \prod_{j\in N} F_j(I) = \sum_{A\in\mathcal{E}} \sum_{|I|>2^k} |I| \prod_{j\in A} \Delta F_j(I) \prod_{j\in N\setminus A} F_j(I).$$

Proof. We compute with the distributive law

$$\prod_{j=1}^n F_j(I_l) = \prod_{j=1}^n (F_j(I) + \Delta F_j(I)) = \sum_{A\subset N} \prod_{j\in A} \Delta F_j(I) \prod_{j\in N\setminus A} F_j(I).$$

$$\prod_{j=1}^n F_j(I_r) = \prod_{j=1}^n (F_j(I) - \Delta F_j(I)) = \sum_{A\subset N} (-1)^{|A|} \prod_{j\in A} \Delta F_j(I) \prod_{j\in N\setminus A} F_j(I).$$

Adding the two identities and bringing the term corresponding to the empty set A to the left hand side and multiplying by $|I|$ or $|I_l|$, $|I_r|$ having factors of 2 in mind, gives

$$-|I|\prod_{j=1}^n F_j(I) + |I_l|\prod_{j=1}^n F_j(I_l) + |I_r|\prod_{j=1}^n F_j(I_r) = \sum_{A\in\mathcal{E}} |I| \prod_{j\in A} \Delta F_j(I) \prod_{j\in N\setminus A} F_j(I).$$

Summing over all $I \in \mathcal{D}_{k'}$ and then summing over all $k < k' \le 0$ proves the theorem. □

We call

$$\Lambda_A(F_1, \dots, F_n) := \sum_{I\in\mathcal{D}} |I| \prod_{i\in A} \Delta F_j(I) \prod_{j\in N\setminus A} F_j(I)$$

the para-product of type A. It is well defined if the infinite sum is absolutely summable over all $\mathcal{D}$,

$$\sum_{I\in\mathcal{D}}|I|\prod_{i\in A}|\Delta F_j(I)|\prod_{j\in N\setminus A}|F_j(I)|<\infty.$$

If all relevant dyadic paraproducts in the last theorem are absolutely summable, we obtain the product formula

$$-\prod_{j\in N}F_j([0,1))+\lim_{k\to-\infty}\sum_{I\in\mathcal{D}_k}|I|\prod_{j\in N}F_j(I)=\sum_{A\in\mathcal{E}}\sum_{I\in\mathcal{D}}|I|\prod_{j\in A}\Delta F_j(I)\prod_{j\in N\setminus A}F_j(I).$$

As products of general Radon measures are not defined, we introduce some smaller subspaces of Radon measures.

Definition 38. *Let m be a Radon measure with martingale extension F. Define*

$$\|m\|_\infty:=\sup_{I\in\mathcal{D}}|F(I)|,$$

$$\|m\|_2:=(|F([0,1))|^2+\sum_{I\in\mathcal{D}}|I||\Delta F(I)|^2)^{1/2}=\lim_{k\to-\infty}(\sum_{I\in\mathcal{D}_k}|I||F(I)|^2)^{1/2}.$$

If the respective quantity is finite, we say that m is in L^∞ or L^2.

The space of Radon measures in L^2 is the Hilbert space identified earlier in terms of Fourier series. The quantity $\|m\|_2^2$ satisfies the parallelogram law and is equal to $\sum_{n\in Z}|\widehat{m}_n|^2$ (Exercise). Any bounded linear map is thus represented by taking the inner product

$$\langle m,m'\rangle=\sum_{I\in\mathcal{D}}|I|\Delta F\overline{\Delta F'}$$

with a certain element in the space.
The space L^∞ is contained in L^2 as we have for each k

$$(\sum_{I\in\mathcal{D}_k}|I||F(I)|^2)^{1/2}\le\|m\|_\infty(\sum_{I\in\mathcal{D}_k}|I|)^{1/2}=\|m\|_\infty.$$

Theorem 39. *Consider notation as in the previous theorem and let m_j for $j\in N$ be Radon measures. Assume m_1 and m_2 are in L^2 and all remaining m_j are in L^∞. Then there is a constant depending only on n such that for each A we have*

$$\sum_{I\in\mathcal{D}}|I|\prod_{i\in A}|\Delta F_j(I)|\prod_{j\in N\setminus A}|F_j(I)|\le C\|m_1\|_2\|m_2\|_2\prod_{j\in N,j>2}\|m_j\|_\infty.$$

We postpone the proof of this theorem. Here we just note the trivial case $\{1,2\}\subset A$, which follows from the observation

$$\sup_{I\in\mathcal{D}}|\Delta F_j(I)|\le\sup_{I\in\mathcal{D}}|F_j(I)|$$

and Cauchy-Schwarz,

$$\sum_{I\in\mathcal{D}}|I|\prod_{i\in A}|\Delta F_j(I)|\prod_{j\in N\setminus A}|F_j(I)|$$

$$\le(\sum_{I\in\mathcal{D}}|I||\Delta F_1(I)|^2)^{1/2}(\sum_{I\in\mathcal{D}}|I||\Delta F_2(I)|^2)^{1/2}\prod_{j\in N,j>2}(\sup_{I\in\mathcal{D}}|F_j(I)|^2)^{1/2}.$$

As a corollary of the last theorem, we obtain the next theorem.

Theorem 40. *Let m_1 be in L^2 and m_2 be in L^∞. Then the limit*

$$m(g) = \lim_{k\to-\infty} \sum_{I\in\mathcal{D}_k} |I| F_1(I)\overline{F_2(I)} g(c(I))$$

exists and defines a Radon measure with $|m(g)| \le \|m_1\|_2 \|m_2\|_\infty \|g\|_2$ where g is identified with the measure of integration against g.

The conclusion implies that the limit defines a Radon measure in L^2.

Proof. Existence of the limit follows by $L^\infty \subset L^2$ and by polarization from Theorem 36. Let G be the martingale extension of integration against g. We have by the observation at the beginning of this section

$$m(g) = \lim_{k\to-\infty} \sum_{k\in\mathcal{D}_k} F_1(I)\overline{F_2(I)} G(I).$$

Decomposing into paraproducts and applying the paraproduct bounds gives

$$|m(f)| \le C\|g\|_2 \|m_1\|_2 \|m_2\|_\infty .$$

□

We note there is an elementary proof of the above corollary without paraproduct decomposition.
We give an example of a further natural and less trivial occurrence of paraproducts. It arises from harmonic analysis on a perturbation of the unit circle as we will elaborate further below.
Let H be a complex valued martingale that is bounded above and below,

$$0 < c \le |H(I)| \le C < \infty$$

for all I. Assume $H([0,1)) = 1$. Assume for simplicity that H is the martingale extension of integration against a function h in $\mathcal{S}^\Delta$. This is a mild assumption, as the estimates below will only depend on h via the constants c and C and in particular not on the minimal scale of the function f.
Define for $I \in \mathcal{D}$ the adapted Haar functions

$$\beta_I(x) = \frac{H(I_r)1_{I_l} - H(I_l)1_{I_r}}{(|I|H(I_l)H(I_r)H(I))^{\frac{1}{2}}} .$$

Note the denominator does not vanish due to the lower bound on H. The square root requires a choice of a sign. We choose a sign, but this choice will be irrelevant for the calculations that follow.
Define for f, g in $\mathcal{S}^\Delta$ the bilinear form

$$\langle f, g\rangle_h = \int_0^1 f(x)g(x)h(x)\,dx.$$

The form is not sesquilinear and not necessarily non-negative for $f = g$.

Theorem 41. *With the definitions as above, we have* $\langle 1,1\rangle_h = 1$*. For any dyadic interval* $I \in \mathcal{D}$ *we have* $\langle \beta_I, 1\rangle_h = 0$ *and* $\langle \beta_I, \beta_I\rangle_h = 1$*. For any further dyadic interval* $J \in \mathcal{D}$ *with* $J \neq I$ *we have* $\langle \beta_I, \beta_J\rangle_h = 0$.

Proof. We have

$$\langle 1,1\rangle_h = \int_0^1 h(x)\,dx = H([0,1)) = 1.$$

Now let $I \in \mathcal{D}$. We have

$$\langle H(I_r)1_{I_l} - H(I_l)1_{I_r}, 1\rangle_h = |I_l|H(I_r)H(I_l) - |I_r|H(I_r)H(1_r) = 0.$$

and therefore $\langle \beta_I, 1\rangle_h = 0$. We have with the martingale identity

$$\langle \beta_I, \beta_I\rangle_h = \frac{|I_l|H(I_r)^2 H(I_l) + |I_r|H(I_l)^2 H(I_r)}{|I|H(I_l)H(I_r)H(I)} = 1$$

Assume further $J \in \mathcal{D}$ with $J \neq I$. If $I \cap J = \varnothing$, then the integrand of the defining integral for $\langle \beta_I, \beta_J\rangle_h$ vanishes, which gives the desired identity. If $I \cap J \neq \varnothing$, we may without loss of generality assume that $I \subset J$ and I is strictly contained in J. Then β_J is constant on the support of I and we have for some constant c

$$\langle \beta_I, \beta_J\rangle_h = c\langle \beta_I, 1\rangle_h = 0.$$

□

Theorem 42. *If* $f \in \mathcal{S}^\Delta$*, then*

$$f = \langle f, 1\rangle 1 + \sum_{I \in \mathcal{D}} \langle f, \beta_I\rangle_h \beta_I,$$

where the sum on the right hand side has finitely many nonzero terms. Moreover, there is a constant C *independent of* f *such that*

$$\frac{1}{C}\|f\|_2^2 \le |\langle f, 1\rangle|^2 + \sum_{I \in \mathcal{D}} |\langle f, \beta_I\rangle_h|^2 \le C\|f\|_2^2.$$

Proof. We may assume $f \in \mathcal{S}_k^\Delta$. The identity follows from the previous theorem, if f is in the span of the function 1 and the functions β_I with $|I| > 2^k$. These are 2^k functions, and they are all in $\mathcal{S}_k^\Delta$. The space $\mathcal{S}_k^\Delta$ has dimension 2^k, so 1 and the β_I span the entire space $\mathcal{S}_k^\Delta$. This proves the first identity.

For the string of inequalities claimed in the theorem, we first prove that the second inequality proves the first. Note that h is bounded above and below because H is bounded below. Hence we may define $g = h^{-1}f$. By Theorem 40, we have that $\|g\|_2 \le C\|f\|_2$ because $\|h\|_\infty$ is bounded by a constant. Then we have with the assumed second inequality for g,

$$\|f\|_2^2 = \langle f, g\rangle_h = \sum_{I \in \mathcal{D}} \langle f, \beta_I\rangle_h \langle \beta_I, g\rangle_h$$

$$\le \left(\sum_{I \in \mathcal{D}} |\langle f, \beta_I\rangle_h|^2\right)^{1/2} \left(\sum_{I \in \mathcal{D}} |\langle g, \beta_I\rangle_h|^2\right)^{1/2} \le C\|g\|_2 \left(\sum_{I \in \mathcal{D}} |\langle f, \beta_I\rangle_h|^2\right)^{1/2}.$$

Estimating $\|g\|_2$ by $C\|f\|_2$ and dividing by $\|f\|_2$ gives the desired first inequality.
To prove the second inequality, we note that by the bounds on $|H(I)|$ it suffices to prove

$$\sum_{I\in\mathcal{D}} |I||H(I_l)G(I_r) - H(I_r)G(I_l)|^2 \le C\|f\|_2^2$$

where G is the martingale extension of $g = fh$. The left hand side equals

$$\sum_{I\in\mathcal{D}} |I||(H(I_l)-H(I_r))(G(I_l)+G(I_r))-(H(I_l)-H(I_r))(G(I_l)+G(I_r))|^2$$

$$= \sum_{I\in\mathcal{D}} |I||\Delta H(I)G(I) - \Delta H(I)G(I)|^2.$$

By the triangle inequality and expanding the square it suffices to prove

$$\sum_{I\in\mathcal{D}} |I|\Delta H(I)\overline{\Delta H(I)}G(I)\overline{G(I)} \le C\|f\|_2^2,$$

$$\sum_{I\in\mathcal{D}} |I|H(I)\overline{H(I)}\Delta G(I)\overline{\Delta G(I)} \le C\|f\|_2^2.$$

But we conclude from Theorem 39 that the left-hand-sides are bounded by

$$\|h\|_\infty^2\|hf\|_2^2 \le C\|f\|_2^2,$$

the latter by Theorem 40. $\square$

The above setup can be interpreted as perturbation of the unit circle. Consider the map

$$\Gamma : \mathbb{T} \to \mathbb{C} \setminus \{0\}$$

defined by

$$\Gamma(\theta) = e^{2\pi i\int_0^\theta h(\sigma)\,d\sigma}.$$

The above assumptions on h are a consequence of the following conditions on Γ (for any $\theta,\theta_2 \in [0,1)$), with different constants c, C, ϵ different but related to the constants in the assumptions on h:

1. *Rectifiability:* $|\Gamma(\theta_1) - \Gamma(\theta_2)| \le C|e^{2\pi i\theta_1} - e^{2\pi i\theta_2}|$.
2. *Chord-arc:* $|\Gamma(\theta_1) - \Gamma(\theta_2)| \ge c|e^{2\pi i\theta_1} - e^{2\pi i\theta_2}|$
3. *Winding number one:* $\int_0^1 \frac{\Gamma'(\theta)}{2\pi i\Gamma(\theta)} = 1$
4. *Distance from zero:* $|\Gamma(\theta_1)| \ge \epsilon$

8.1. **Exercise.** Let $F : \mathcal{D} \to \mathbb{C}$ satisfy the martingale identity.
a)Prove that if $\sup_I |F(I)| < \infty$, then F is the martingale extension of an absolutely continuous Radon measure.
b) Prove that if $\sum_I |I||\Delta F(I)|^2 < \infty$, then m is the martingale extension of an absolutely continuous Radon measure.

8.2. **Exercise.** Let m be a Radon measure on $[0,1)$ and F its martingale extension. Then we have

$$\sum_{n\in\mathbb{Z}} |\widehat{m}(n)|^2 = \sum_{I\in\mathcal{D}} |F(I)|^2$$

in the sense that if one of the two sides is finite, then so is the other and both sides are equal.

9. Outer measures and their L^p theory

Our goal is to obtain upper bounds on absolute sums such as those associated with paraproducts,

$$\sum_{I\in\mathcal{D}} |I| \prod_{i\in A} |\Delta F_j(I)| \prod_{j\in N\setminus A} |F_j(I)|.$$

For convenience, we will consider finite subsums. Define

$$\mathcal{D}' = \{I \in \mathcal{D} : |I| \geq 2^{-N}\}$$

for some large N. Of interest will be estimates for analoguous sums as above over $\mathcal{D}'$ with constants not depending on N. Then a limiting process will yield the same upper bounds for the full inifinite sum.

We seek norms on functions $F : \mathcal{D}' \to \mathbb{C}$. Our approach will be utilizing an outer measure on $\mathcal{D}'$. For each $I \in \mathcal{D}'$ define the tree T_I to be the set of all intervals $J \in \mathcal{D}'$ with $J \subset I$. Let $\mathcal{E}$ be the set of all trees and define $\sigma : \mathcal{E} \to (0,\infty)$ by $\sigma(T_I) = |I|$.

For a subset $A \subset \mathcal{D}'$, define the outer measure of A to be

$$\mu(A) := \inf_{\mathcal{E}'\subset\mathcal{E} : A \subset \bigcup \mathcal{E}'} \sum_{T\in\mathcal{E}'} \sigma(T).$$

We call a collection $\mathcal{E}'$ as in the infimum of this definition a cover of A. As we are working on a finite set $\mathcal{D}'$, the infimum will be attained by some cover. We have $\mu(\emptyset) = 0$, because the empty collection covers the empty set. Moreover, we have the sub additivity property that for two subsets $A, B \subset \mathcal{D}'$

$$\mu(A \cup B) \leq \mu(A) + \mu(B), \tag{4}$$

because for any two covers $\mathcal{E}'$ and $\mathcal{E}''$ of respectively A and B, the union $\mathcal{E}' \cup \mathcal{E}''$ covers $A \cup B$.

For each tree T_I we have $\sigma(T_I) = \mu(T_I)$. The inequality $\sigma(T) \geq \mu(T)$ holds, because the set $\mathcal{E}' = \{T_I\}$ covers T_I. The inequality $\sigma(T) \leq \mu(T)$ holds, because any cover $\mathcal{E}'$ of T_I needs to satisfy $I \in \bigcup \mathcal{E}'$ and thus there needs to be a member $T_J \in \mathcal{E}'$ with $I \subset J$.

The inequality in (4) may be strict, even if A and B are disjoint. The disjoint sets $A = \{I\}$ and $B = \{I_l\}$ for example satisfy similarly to the previous arguments $\mu(A) = |I|$ and $\mu(B) = |I|/2$ and $\mu(A \cup B) = |I|$.

For a function $F : \mathcal{D}' \to \mathbb{C}$ we consider "local" norms

$$\ell^\infty F(T) = \sup_{J\in T} |F(J)|$$

and for $1 \leq q < \infty$

$$\ell^q F(T) = \left(\frac{1}{\sigma(T)} \sum_{J\in T} |J||F(J)|^q\right)^{1/q}, \tag{5}$$

which can alternatively be expressed as

$$\sigma(T)(\ell^q F(T))^q = \sum_{J\in T} |J||F(J)|^q = \sum_{J\in T} \int_0^{|F(J)|^q} |J|\, d\lambda$$

$$= \int_0^\infty \sum_{J\in T:|F(J)|^q>\lambda} |J|\, d\lambda = \int_0^\infty \inf\{\sum_{J\in A} |J| : A \subset T, \ell^\infty(F1_A^c) \le \lambda^{1/q}\}\, d\lambda.$$

This is the socalled layer cake representation.
We recall the classical subadditivity and submultiplicativity (Hölder) of these norms. For any $1 \le q \le \infty$

$$\ell^q(\sum_{i=1}^n F_i)(T) \le \sum_{i=1}^n \ell^q F_i(T).$$

Let $1 \le q, q_i \le \infty$ such that $\frac{1}{q} = \sum_{i=1}^n \frac{1}{q_i}$. Then

$$\ell^q(\prod_{i=1}^n F_i)(T) \le \prod_{i=1}^n \ell^{q_i} F_i(T).$$

The submultiplicativity underlines the importance to work with a family of norms parameterized by q.
The norms (5) are used to define a "global" L^∞ norm, that is for all $1 \le q \le \infty$

$$L^\infty \ell^q F = \sup_{T\in\mathcal{E}} \ell^q F(T).$$

and for $1 \le p < \infty$ in analogy to the above layer cake representation

$$L^p \ell^q F = (\int_0^\infty \inf\{\mu(A) : L^\infty \ell^q(F1_{A^c}) \le \lambda^{1/p}\}\, d\lambda)^{1/p}.$$

We do not call this quantity a norm, as it merely satisfies a quasi triangle inequality.

Theorem 43. *For any* $1 \le p, q \le \infty$

$$L^p \ell^q (F + G) \le 2(L^p \ell^q F + L^p \ell^q G).$$

If $p = \infty$*, the constant* 2 *may be replaced by* 1*.*

Proof. If $p = \infty$, this follows by the norm property of ℓ^q for fixed $T \in \mathcal{E}$. Assume $p < \infty$. Consider $\lambda > 0$ and let B and C be of minimal outer measure such that

$$L^\infty \ell^q(F1_{B^c}) \le \lambda^{1/p},$$
$$L^\infty \ell^q(G1_{C^c}) \le \lambda^{1/p}.$$

Then by monotonicity of the norm $L^\infty \ell^q$

$$L^\infty \ell^q((F+G)1_{(B\cup C)^c}) \le L^\infty \ell^q(F1_{(B\cup C)^c}) + L^\infty \ell^q(G1_{(B\cup C)^c})$$
$$\le L^\infty \ell^q(F1_{B^c}) + L^\infty \ell^q(G1_{C^c}) \le 2\lambda^{1/p}.$$

Letting λ vary, we write B_λ and C_λ for the above sets.

$$L^p \ell^q (F+G)^p \le \int_0^\infty \inf\{\mu(A) : L^\infty \ell^q((F+G)1_{A^c}) \le \lambda^{1/p}\}\, d\lambda.$$
$$\le 2^p \int_0^\infty \inf\{\mu(A) : L^\infty \ell^q((F+G)1_{A^c}) \le 2\lambda^{1/p}\}\, d\lambda.$$
$$\le 2^p \int_0^\infty \mu(B_\lambda \cup C_\lambda)\, d\lambda$$
$$\le 2^p((L^p \ell^q F)^p + (L^p \ell^q G)^p) \le 2^p(L^p \ell^q F + L^p \ell^q G)^p.$$

The desired inequality follows by taking p-th roots. $\square$

Theorem 44 (Hölder's inequality with constant)**.** *Let* $1 \le p, p_1, \dots, p_n \le \infty$ *such that* $\frac{1}{p} = \sum_{i=1}^n \frac{1}{p_i}$. *Let* $1 \le q, q_1, \dots, q_n \le \infty$ *such that* $\frac{1}{q} = \sum_{i=1}^n \frac{1}{q_i}$. *Then we have*

$$L^p \ell^q (\prod_{i=1}^n F_i) \le n^{1/p} \prod_{i=1}^n L^{p_i} \ell^{q_i} F_i.$$

Proof. Assume first that $p = \infty$ and thus $p_i = \infty$ for all i and thus also $p = \infty$. Set $F = \prod_{i=1}^n F_i$. Then

$$L^\infty \ell^q F = \sup_{T \in \mathcal{E}} \ell^q F(T) \le \sup_{T \in \mathcal{E}} \prod_{i=1}^n \ell^{q_i} F_i(T) \le \prod_{i=1}^n \sup_{T \in \mathcal{E}} \ell^{q_i} F_i(T) \le \prod_{i=1}^n L^\infty \ell^{q_i} F_i.$$

Consider now general $p < \infty$. By dividing each F_i by its norm, we may assume that for each i

$$L^{p_i} \ell^{q_i} F_i = 1.$$

Fix $\lambda > 0$ and pick covers A_i such that $A_i = \varnothing$ if $p_i = \infty$ and otherwise

$$L^\infty \ell^{q_i} (F_i 1_{A_i^c}) \le \lambda^{1/p_i}$$

and

$$\mu(A_i) = \inf\{\mu(A) : L^\infty \ell^{q_i} (F_i 1_{A^c}) \le \lambda^{1/p_i}\}.$$

Define $A = \bigcup_{i=1}^n A_i$. We obtain with the proven case of $L^\infty \ell^q$

$$L^\infty \ell^q (F 1_{A^c}) \le L^\infty \ell^q (\prod_{i=1}^n F_i 1_{A_i^c}) \le \prod_{i=1}^n L^\infty \ell^{q_i} (F_i 1_{A_i^c}) \le \prod_{i=1}^n \lambda^{1/p_i} = \lambda^{1/p}.$$

Letting λ vary and denoting the sets above by A_λ and $A_{i,\lambda}$ we obtain

$$(L^p \ell^q F)^p \le \int_0^\infty \mu(A_\lambda)\, d\lambda \le \sum_{i=1}^n \int_0^\infty \mu(A_{i,\lambda}) \le n.$$

$\square$

Let $F : \mathcal{D}' \to \mathbb{C}$ and $1 \le q \le \infty$ be given. We choose a sequence of trees T_i, $i = 1, 2, \dots$ by the following greedy procedure. Assume we have already chosen T_j for $j < i$ and let

$$A_i = \bigcup_{j<i} T_j.$$

If $L^\infty \ell^q (F 1_{A_i^c}) = 0$, we stop the selection. Otherwise, we pick a tree T_i so that

$$\ell^q (F 1_{A_i^c})(T_i)$$

is maximal. Thanks to finiteness of $\mathcal{D}'$, such a tree exists, though it may not be unique.

In what follows, we shall not write the index i. If T is a selected tree, we write A_T for the union of trees selected prior to the considered tree T. Let $\mathcal{E}_\kappa$ be the set of selected trees T such that

$$2^\kappa < \ell^q (F 1_{A_T^c})(T) \le 2^{\kappa+1}.$$

Let A_κ be the union of all trees chosen prior to all trees in $\mathcal{E}_\kappa$.

Theorem 45. *Let $1 \le q < \infty$. With a greedy choice of trees as above, we have for all κ*

$$\sum_{T\in\mathcal{E}_\kappa} \sigma(T) \le C \inf\{\mu(A) : L^\infty \ell^q(F 1_{A^c}) > 2^{\kappa-1}\}$$

with some constant C depending only on q.

Proof. Fix κ and let A be a set attaining the infimum on the right hand side of the desired inequality. Let $\mathcal{E}_A$ be a cover of A attaining the outer measure, that is

$$\mu(A) = \sum_{T\in\mathcal{E}_A} \sigma(T).$$

We have for each $T \in \mathcal{E}_\kappa$ by definition of $\mathcal{E}_k$

$$\sum_{J\in T\setminus A_T} |J||F(J)|^q \ge 2^{q\kappa}\sigma(T)$$

and by definition of A

$$\sum_{J\in T\setminus(A\cup A_T)} |J||F(J)|^q \le 2^{q(\kappa-1)}\sigma(T).$$

Hence

$$\sum_{J\in(A\cap T)\setminus A_T} |J||F(J)|^q \ge 2^{q\kappa}(1-2^{-q})\sigma(T).$$

Now we estimate with the greedy information about A_κ,

$$\sum_{T\in\mathcal{E}_A} 2^{q(\kappa+1)}\sigma(T) \ge \sum_{T\in\mathcal{E}_A}\sum_{J\in T\setminus A_\kappa} |J||F(J)|^q$$

$$\ge \sum_{J\in A\setminus A_\kappa} |J||F(J)|^q \ge \sum_{T\in\mathcal{E}_\kappa}\sum_{J\in(A\cap T)\setminus A_T} |J||F(J)|^q \ge \sum_{T\in\mathcal{E}_\kappa} 2^{qk}(1-2^{-q})\sigma(T).$$

This completes the proof of the Theorem. ∎

Theorem 46. *For $1 \le p < \infty$ there is a constant C such that for all $F\mathcal{D}' \to \mathbb{C}$*

$$\frac{1}{C}\sum_{\mathcal{D}} |I||F(I)|^p \le (L^p\ell^p F)^p \le C\sum_{\mathcal{D}} |I||F(I)|^p$$

Note that an analoguous statement for $p = \infty$ is trivial,

$$\sup_{J\in\mathcal{D}} |F(J)| = L\infty\ell^\infty F.$$

Proof. Constants C depend on p and may change from step to step. Bounding a monotone integrand by a step function gives

$$(L^p\ell^p F)^p \le C\sum_{\kappa\in\mathbb{Z}} 2^{p\kappa}\inf\{\mu(A) : L^\infty\ell^p(F1_{A^c}) \le 2^\kappa\}.$$

By the greedy procedure and then by Fubini and a geometric series:

$$\le C\sum_{\kappa\in\mathbb{Z}} 2^{p\kappa}\sum_{\mu\ge\kappa}\sum_{T\in\mathcal{E}_\mu}\sigma(T) \le C\sum_{\mu\in\mathbb{Z}}\sum_{T\in\mathcal{E}_\mu}\sigma(T) \le C\sum_{\mu\in\mathbb{Z}}\sum_{T\in\mathcal{E}_\mu}\sum_{J\in T\setminus A_T} |J||F(J)|^p$$

The last expression is equal to

$$C \sum_{J \in \mathcal{D}'} |J||F(J)|^p$$

proving one of the claimed inequalities. Conversely, we have with the previous theorem and the same expansion for the last display

$$\sum_{\mu\in\mathbb{Z}} \sum_{T\in\mathcal{E}_\mu} \sum_{J\in T\setminus A} |J||F(J)|^p \le \sum_{\mu\in\mathbb{Z}} 2^{p(\mu+1)} \sum_{T\in\mathcal{E}_\mu} \sigma(T).$$

$$\le C \sum_{\mu\in\mathbb{Z}} 2^{p(\mu+1)} \inf\{\mu(A) : L^\infty \ell^q(F1_{A^c}) > 2^{\mu-1}\} \le C(L^p\ell^p F)^p.$$

□

For $1 \le p \le \infty$ denote with p' the dual exponent, $\frac{1}{p} + \frac{1}{p'} = 1$ or equivalently $p' = \frac{p}{p-1}$.

Theorem 47. *Let $1 \le p < \infty$ and $1 \le q < \infty$. For each F with $L^p\ell^q F = 1$ there is a G with $L^{p'}\ell^{q'} G \le C$ and $L^1\ell^1(FG) \ge 1/C$.*

Proof. Consider again the greedy selection of trees. Define

$$G = \sum_{\kappa\in\mathbb{Z}} 2^{(p-q)\kappa} 1_{A_{\kappa-1}\setminus A_\kappa} \overline{F}|F|^{q-2}$$

If F vanishes at some point I, we set $\overline{F}|F|^{p-2} = 0$ at this point. We claim that for every κ

$$L^\infty \ell^{q'}(G1_{A_\kappa^c}) \le C2^{\kappa(p-1)}.$$

For $q = 1$, this follows form pointwise estimation of G. To see the claim for $q > 1$, let T be any tree. Then

$$\sigma(T)\ell^{q'}(G1_{A_\kappa^c})(T)^{q'} = \sum_{J\in T\setminus A_\kappa} |J||G(J)|^{q'}$$

$$\le \sum_{\mu\le\kappa} \sum_{J\in(T\cap A_{\mu-1})\setminus A_\mu} |J|2^{\mu(q'(p-1)-q)}|F(J)|^q$$

$$\le C \sum_{\mu\le\kappa} 2^{\mu q'(p-1)}\sigma(T) \le C2^{\kappa q'(p-1)}\sigma(T).$$

This proves the claim. For $p = 1$ it shows immediately $L^{p'}\ell^{q'}G \le C$ by choosing κ small enough. For $p > 1$ we have

$$(L^{p'}\ell^{q'})G^{p'} \le C \sum_\kappa 2^{p'\kappa(p-1)} \inf\{\mu(A) : L^\infty\ell^{q'}G \ge C2^{\kappa(p-1)}\}$$

$$\le C \sum_\kappa 2^{p\kappa} \sum_{\mu>\kappa} \sum_{T\in\mathcal{E}_\mu} \sigma(T)$$

$$\le C \sum_\kappa 2^{p\kappa} \sum_{\mu>\kappa} \inf\{\mu(A) : L^\infty\ell^q F \ge 2^{\mu-1}\}$$

$$\le C \sum_\mu 2^{p\mu} \inf\{\mu(A) : L^\infty\ell^q F \ge 2^{\mu-1}\} \le C(L^p\ell^q F)^p \le C$$

On the other hand, we have

$$1 \le (L^p\ell^q F)^p \le C \sum_{\kappa\in\mathbb{Z}} 2^{p\kappa} \inf\{\mu(A) : L^\infty\ell^q(F1_{A^c}) \ge C2^\kappa\}$$

$$\leq C \sum_{\kappa\in\mathbb{Z}} 2^{p\kappa} \sum_{\mu>\kappa} \sum_{T\in\mathcal{E}_\mu} \sigma(T) \leq C \sum_{\mu\in\mathbb{Z}} \sum_{T\in\mathcal{E}_\mu} 2^{p\mu}\sigma(T)$$

$$\leq C \sum_{\mu\in\mathbb{Z}} \sum_{T\in\mathcal{E}_\mu} \sum_{J\in T\setminus A_T} 2^{(p-q)\mu}|J||F(J)|^q$$

$$C \sum_J |J||F(J)G(J)| \leq CL^1\ell^1(FG).$$

This proves the other claimed inequality and completes the proof of the theorem. □

We notice that the quantity $L^p\ell^q F$ is up to constants equivalent to

$$\left| \sup_{G:L^{p'}\ell^{q'}G\leq 1} \sum_{J\in\mathcal{D}'} F(J)G(J) \right|$$

This quantity however is a norm and in particular satisfies subadditivity.
We comment on variants of Theorem 47 in the instance that p or q are infinite. If both are infinite, Theorem 46 serves as good replacement of Theorem (47).
If only q is infinite, we define $A_\kappa^{\max}$ the set of maximal dyadic intervals in $A_{\kappa-1}\setminus A_\kappa$ such that $|F(I)|\geq 2^\kappa$. Then we define

$$G = \sum_{\kappa\in\mathbb{Z}} 2^{(p-1)\kappa} \sum_{I\in A_\kappa^{\max}} 1_{\{I\}}\overline{F}|F|^{-1}$$

We see similarly to above for every tree T:

$$\sigma(T)\ell^1(G1_{A_\kappa^c})(T) = \sum_{J\in T\setminus A_\kappa} |J||G(J)|$$

$$\leq \sum_{\mu\leq\kappa} \sum_{J\in T\cap A_\mu^{\max}} |J|2^{\mu(p-1)} \leq C\sum_{\mu\leq\kappa} 2^{\mu(p-1)}\sigma(T) \leq C2^{\kappa(p-1)}\sigma(T).$$

where we have used that the intervals in $T\cap A_\mu^{\max}$ are disjoint and contained in the top interval of T of length $\sigma(T)$. With this we proceed as in the proof of Theorem 47.
If only p is infinite, we identify a tree T such that

$$1 = L^\infty\ell^q F = \ell^q F(T)$$

and define

$$G = \sigma(T)^{-1}1_T\overline{F}|F|^{q-2}.$$

We note

$$\ell^{q'}G(T) = \sigma(T)^{-1}.$$

Then we note

$$L^1\ell^{q'}G \leq \int_0^{\sigma(T)^{-1}} \sigma(T)\,d\lambda = 1.$$

And

$$\sum_J |J|F(J)G(J) = \sigma(T)^{-1}\sigma(T)\ell^q F(T) = 1.$$

10. Lecture: Embedding theorem, dyadic BMO and H^1

(Tuesday May 26. 2020)
Consider a paraproduct of type $A \subset N$. The set A contains at least two elements, by symmetry we assume without loss of generality that $\{1,2\} \subset A$. We obtain with Hölder's inequality for outer L^p norms

$$\sum_{I\in\mathcal{D}} |I| \prod_{i\in A} |\Delta F_j(I)| \prod_{j\in N\setminus A} |F_j(I)|$$

$$\le (L^{p_i}\ell^2 \Delta F_1)(L^{p_2}\ell^2\Delta F_1) \prod_{j>2} (L^{p_j}\ell^\infty F_1),$$

where we use the estimate

$$L^p\ell^\infty \Delta F \le L^p \ell^\infty F,$$

which follows from

$$\ell^\infty \Delta F(T) \le \ell^\infty F(T)$$

for arbitrary tree T. Assuming F_j is the martingale extension of m_j , the next theorem will control the outer L^p norms by the norms

$$\|m\|_p := \sup_{k\le 0} (\sum_{I\in\mathcal{D}_k} |I||F(I)|^p)^{1/p} = \lim_{k\to-\infty} (\sum_{I\in\mathcal{D}_k} |I||F(I)|^p)^{1/p}$$

for $1 \le p < \infty$ and

$$\|m\|_\infty = \sup_{I\in\mathcal{D}} |F(I)|.$$

Theorem 48. *[Embedding theorem] For $1 < p \le \infty$ and F the martingale extension of m we have for some constant C depending only on p:*

$$L^p\ell^\infty F \le C\|m\|_p, \tag{6}$$

$$L^p\ell^2\Delta F \le C\|m\|_p. \tag{7}$$

Proof. We prove the theorem in three steps. The first step consists of verifying the theorem in case $p = \infty$. We have

$$L^\infty\ell^\infty F = \sup_{T\in\mathcal{E}} \sup_{J\in T} |F(J)| = \sup_{J\in\mathcal{D}} |F(J)| = \|m\|_\infty$$

and

$$(L^\infty\ell^2\Delta F)^2 = \sup_{T\in\mathcal{E}} \sigma(T)^{-1} \sum_{J\in T} |J||\Delta F(J)|^2$$

$$\le \sup_{T\in\mathcal{E}} \sigma(T)^{-1} \sup_k \sum_{J\in\mathcal{D}_k : J\in T} |J||F(J)|^2$$

$$\le \|m\|_\infty^2 \sup_{T\in\mathcal{E}} \sup_k \sigma(T)^{-1} \sum_{J\in\mathcal{D}_k : J\in T} |J| \le \|m\|_\infty^2.$$

Here we have from the first to the second line used the telescoping identity. namely for every tree T_I

$$\sum_{J\in T_I} |J||\Delta F(J)|^2 = -|I||F(I)|^2 + \lim_{k\to-\infty} \sum_{J\in\mathcal{D}_k : J\in T_I} |J||F(J)|^2.$$

The second step of the proof establishes weak endpoint bounds at $p = 1$, namely for all $\lambda > 0$

$$\lambda \inf\{\mu(A) : L^\infty \ell^\infty (F 1_{A^c}) \le \lambda\} \le \|m\|_1, \tag{8}$$

$$\lambda \inf\{\mu(A) : L^\infty \ell^2 (\Delta F 1_{A^c}) \le \lambda\} \le 2\|m\|_1. \tag{9}$$

To see (8), let $\lambda > 0$ and let $\mathcal{E}'$ be the set of all trees T_I such that I is a maximal dyadic interval with respect to set inclusion with $|F(I)| > \lambda$. Since every J with $|F(J)| > \lambda$ is contained in a maximal J' with $|F(J')| > \lambda$ and thus $J \in \bigcup \mathcal{E}'$, we have

$$L^\infty \ell^\infty F 1_{(\bigcup \mathcal{E}')^c} \le \lambda.$$

Hence the left hand side of (8) is bounded by

$$\lambda \sum_{T \in \mathcal{E}'} \sigma(T) \le \sum_{T_I \in \mathcal{E}'} |I||F(I)|.$$

To estimate the last sum, it suffices to estimate sums over arbitrary finite $\mathcal{E}'' \subset E$. By the martingale identity and the triangle inequality we have for each T_I and each $2^k < |I|$,

$$|I||F(I)| \le \sum_{J \in T_I \cap \mathcal{D}_k} |J||F(J)|.$$

Picking 2^k smaller than the minimum of $|I|$ with $T_I \in \mathcal{E}''$ we obtain with disjointness of these I

$$\lambda \sum_{T \in \mathcal{E}'} \sigma(T) \le \sum_{J \in \mathcal{D}_k} |J||F(J)| \le \|m\|_1.$$

This completes the proof of (8).

To see (9), let $\mathcal{E}^*$ be the set of all trees T_I such that I is maximal with respect to set inclusion among the dyadic intervals satisfying

$$\max(|F(I_l)|, |F(I_r)|) > \lambda.$$

As each $I \in \mathcal{E}^*$ is associated with a corresponding child in $\mathcal{E}'$ of half the length, we obtain

$$\lambda \sum_{T \in \mathcal{E}*} \sigma(T) \le 2\lambda \sum_{T \in \mathcal{E}'} \sigma(T) \le 2\|m\|_1.$$

For $J \in \mathcal{D}$ define $G(J) = F(I)$ if $J \in T_I$ for some $T_I \in \mathcal{E}^*$ and $G(J) = F(J)$ otherwise. Set $A = \bigcup \mathcal{E}^*$. If $J \in A$, then

$$G(J) = G(J_l) = G(J_r),$$

while if $J \notin A$ then

$$G(J) = F(J), G(J_l) = F(J_l), G(J_r) = F(J_r),$$

because if $J \notin A$ and say $J_l \in A$ then $J_l = I$ for some $T_I \in \mathcal{E}^*$ and similarly for J_r. In particular, G is a martingale. Moreover, if $J \in A$, then $\Delta F(J) = 0$ and if $J \notin A$, then $\Delta G(J) = \Delta F(J)$. We obtain

$$L^\infty \ell^2 (\Delta F 1_{A^c}) = L^\infty \ell^2 (\Delta G) \le \lambda,$$

the last inequality by the first step of the proof and the fact that G is bounded by λ. Hence

$$\lambda \inf\{\mu(A) : L^\infty \ell^2(\Delta F 1_{A^c}) \le \lambda\} \le \lambda \sum_{T\in\mathcal{E}^*} \sigma(T) \le 2\|m\|_1.$$

This completes the second step of the proof. The decomposition

$$F = G + (F - G)$$

in the above argument is called the dyadic Calderón Zygmund decomposition of F at level λ.

The third step interpolates between the first two steps, this is called Marcinkiewicz interpolation.

We show (7). It suffices to prove the estimate for m being integration against a function $f \in \mathcal{S}_k^\Delta$ for arbitrary k. For $\lambda > 0$ and split

$$f = g_\lambda + h_\lambda$$

with

$$g_\lambda = \sum_{I\in\mathcal{D}_k : |F(I)|\le\lambda} F(I) 1_I .$$

Let G_λ and H_λ be the corresponding martingales. By the L^∞ bound of the first step, we have for any A,

$$L^\infty \ell^2(\Delta G_\lambda 1_{A^c}) \le \lambda$$

because g and hence ΔG_λ is bounded by λ. Hence

$$\inf\{\mu(A) : L^\infty \ell^2(\Delta F 1_{A^c}) > 2\lambda\}$$
$$\le \inf\{\mu(A) : L^\infty \ell^2(\Delta H_\lambda 1_{A^c}) > \lambda\} \le 2\lambda^{-1}\|h\|_1,$$

the last inequality by the weak bound at $p = 1$ from the second step. We have

$$\|h\|_1 = \sum_{I\in\mathcal{D}_k : |F(I))|>\lambda} |I||F(I)| \le 2^{k+1} \sum_{I\in\mathcal{D}_k} \int_{\lambda/2}^{|F(I)|} d\nu$$
$$\le 2^{k+1} \int_{\lambda/2}^\infty |\{I \in \mathcal{D}_k : |F(I)| \ge \nu\}| \, d\nu,$$

where the integrand in the last integral is the cardinality of the set of intrvals I with the described property. Hence we have

$$(L^p \ell^2 \Delta F)^p \le C \int_0^\infty \lambda^{p-1} \inf\{\mu(A) : L^\infty \ell^2(\Delta F 1_{A^c}) > 2\lambda\} \, d\lambda$$
$$\le C \int_0^\infty \lambda^{p-2} \int_{\lambda/2}^\infty |\{I \in \mathcal{D}_k : |F(I)| \ge \nu\}| \, d\nu \, d\lambda$$
$$\le C \int_0^\infty |\{I \in \mathcal{D}_k : |F(I)| \ge \nu\}| \int_0^{2\nu} \lambda^{p-2} \, d\lambda \, d\nu$$
$$\le C \int_0^\infty \nu^{p-1} |\{I \in \mathcal{D}_k : |F(I)| \ge \nu\}| \, d\nu \le C \sum_{I\in\mathcal{D}_k} |I||F(I)|^p.$$

The proof for the inequality using martingale averages is done analogously. This completes the proof of the third step and the theorem. $\square$

The next theorem shows that for $1 < p < \infty$ the quantities $\|m\|_p$ and $L^p\ell^2\Delta F$ are equivalent for F the martingale extension of m and thus define the same space, with small care to be taken for the measure given by integration against a constant function. In particular, while $L^p\ell^2\Delta F$ may not irself be a norm, it is equivalent to a norm.

Theorem 49. *For $1 < p < \infty$ we have the reverse embedding inequality: for every measure m such that $m(1) = 0$*

$$\|m\|_p \le CL^p\ell^2\Delta F.$$

Proof. It suffices to prove this for m being integration against $f \in \mathcal{S}^\Delta$ with $\int_0^1 f(x)dx = 0$ and $\|f\|_p = 1$. With $g = \overline{f}|f|^{p-2}$ we obtain with the paraproduct decomposition and Hölder

$$1 = \|f\|_p^p = \int_0^1 f(x)g(x)\,dx = \sum_{J\in\mathcal{D}} |J|\Delta F(J)\Delta G(J)$$
$$\le (L^p\ell^2\Delta F)(L^{p'}\ell^2\Delta G) \le C\|g\|_{p'}L^p\ell^2\Delta F = CL^p\ell^2 F$$

□

For $p = \infty$, it turns out that the embedding inequality cannot be reversed and the space of bounded martingales is not the same as the space of martingales F with finite $L^\infty\ell^2\Delta F$.

Definition 50. *We call a measure m if dyadic bounded mean oscillation , in short dyadic BMO, if its martingale extension F satisfies*

$$L^\infty\ell^2\Delta F < \infty.$$

We note that the dyadic BMO is not tranlastion invariant under translation of the periodic measure. Genuine BMO is the space of measures such that all its translates are in dyadic BMO. For the time being we work with dyadic BMO only.

Theorem 51. *There is a measure in dyadic BMO that is not in L^∞.*

Proof. We consider a martingale F with $F([0,1)) = 0$ such that $\Delta F(J) = 1$ if the left endpoint of J is zero and $\Delta F(J) = 0$ otherwise. Then

$$\sum_J |J||\Delta F(J)|^2 = \sum_{k\le 0}\sum_{J\in\mathcal{D}_k} |J||\Delta F(J)|^2 \le \sum_{k\le 0} 2^{-k} = 2.$$

In particular, F comes from a Radon measure in L^2. Moreover, for every interval I of length 2^l

$$\ell^2\Delta F(T_I) = |I|^{-1}\sum_{J\in T_I} |J||\Delta F(J)|^2$$
$$= |I|^{-1}\sum_{k\le 0}\sum_{J\in\mathcal{D}_k\cap T_I} |J||\Delta F(J)|^2 \le 2^{-l}\sum_{k\le l} 2^k = 2.$$

Hence $L^\infty\ell^2\Delta F \le 2$. On the other hand, we have

$$F([0,1)) = 0$$
$$F([0,2^{k-1})) = F([0,2^k)) + 1.$$

By induction,

$$F([0,2^k)) = -k.$$

Hence F is not bounded.

□

The logarithmic blowup in the previous theorem is typical for BMO. Note that every measure in BMO is in L^p for all $p < \infty$, not allowing for any polynomial blow-up.
The situation at the other endpoint $p = 1$ is similar but reverse.

Definition 52. *We say a Radon measure with $m(1) = 0$ is in the dyadic Hardy space H^1, if its martingale satisfies*

$$L^1\ell^2\Delta F < \infty.$$

Theorem 53. *There is a Radon measures m with $m(1) = 0$ which is not in the dyadic Hardy space H^1*

Proof. We take the Radon measure $m(f) = f(0)$. We have for its martingale extension

$$F([0,2^k)) = 2^{-k}$$

and $F(J) = 0$ if J does not contain 0. Hence also

$$\Delta F([0,2^k)) = 2^{-k}.$$

Hence every cover $\mathcal{E}'$ such that

$$L^\infty\ell^2(\Delta F 1_{(\bigcup\mathcal{E}')^c}) \leq 2^{-k}$$

must cover the interval $[0,2^{k-1})$ and hence

$$\inf\{\mu(A) : L^\infty\ell^2(F1_{(\bigcup\mathcal{E}')^c}) \leq 2^{-k}\} \geq 2^{k-1}.$$

But then,

$$L^1\ell^2\Delta F \geq \sum_{k=0}^{\infty} 2^k \inf\{\mu(A) : L^\infty\ell^2(\Delta F 1_{(\bigcup\mathcal{E}')^c}) \leq 2^k\} \geq \sum_{k=0}^{\infty} 2^{-1} = \infty.$$

□

The situation does not change in this theorem if we additionally require m to be absolutely continuous. Pick an absolutely summable sequence a_n and pick k_n rapidly shrinking in dependence of a_n and consider

$$m(f) = \sum_{n=1}^{\infty} a_n m_n(f),\ m_n(f) = \int_0^{2^{k_n}} f(x)\,dx$$

Thanks to summability of a_n we obtain an absolutely continuous measure, since absolute continuity is closed under convergence in total mass (Exercise). The measures m_n approximates the Dirac delta and thus has Hardy space norm grows very rapidly for suitable sequence k_n. Hence m fails to be in the dyadic Hardy space.

11. Lecture: Cauchy integral

(Thursday May 28. 2020)
Consider a closed curve Γ in the complex plane in the form of the image of a parameterization

$$\gamma : \mathbb{R} \to \mathbb{C}$$

which is one- periodic, $\gamma(t) = \gamma(t+1)$. We assume γ is Lipschitz, that is

$$|\gamma(t) - \gamma(s)| \le C|t-s|$$

for all $t, s \in \mathbb{R}$, and we assume it is chord-arc, that is

$$c|t-t| \le |\gamma(t) - \gamma(s)|$$

for all $t, s \in \mathbb{R}$. In particular,

$$G(I) = \gamma(r(I)) - \gamma(l(I))$$

defines a martingale that is bounded in absolute value above by C and below by c.
We assume for the moment that γ is is the primitive of a function $\gamma' \in \mathcal{S}^{\Delta}$, then G is the martingale extension of γ'. Consider a function f on the curve so that $f \circ \gamma$ is continuous with martingale extension F. The Cauchy integral of f at a point z not in the range of the curve is then defined as

$$C_\Gamma f(z) = \frac{1}{2\pi i} \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t) dt$$
$$= \frac{1}{2\pi i} \lim_{k \to -\infty} \sum_{I \in \mathcal{D}_k} |I| G(I) F(I) H(I),$$

where H is the martingale extension of the continuous function

$$\frac{1}{\gamma(t) - z}.$$

Thanks to paraproduct estimates, the martingale expression continues to make sense for example if F is in $L^p\ell^2$ and G is in $L^{p'}\ell^2 G$ for $1 < p < \infty$, and H is in $L^\infty \ell^\infty H$. The last two conditions follow from γ Lipschitz.
A particular instance of the Cauchy integral is $f = 1$, when it is called the winding number of the curve about the point z.

Theorem 54. *The winding number $C_\Gamma 1(z)$ for z not on the curve Γ is a natural number.*

Proof. By approximation, it suffices to consider the case of $\gamma' \in \S^d$ and $f \circ \gamma$ continuous. Consider the functions

$$g(t) = \exp(\int_0^t \frac{1}{\gamma(s) - z} \gamma'(s) ds),$$
$$h(t) = \frac{g(t)}{\gamma(t) - z}.$$

The function h is constant, since

$$g'(t) = \frac{1}{\gamma(t) - z}\gamma'(t)g(t),$$

$$h'(t) = -\frac{g(t)\gamma'(t)}{(\gamma(t)-z)^2} + \frac{g'(t)}{\gamma(t)-z} = 0.$$

Equating the values at $t = 1$ and $t = 0$ gives $\exp(2\pi i C_\Gamma 1(z)) = 1$ and hence $C_\Gamma 1(z) = n$ for some integer n. □

The winding number of the unit circle

$$\gamma(t) = \exp(2\pi i t)$$

about the point $z = 0$ is 1. Namely

$$C_\Gamma 1(0) = \frac{1}{2\pi i}\int_0^1 \frac{1}{\exp(2\pi i t)} 2\pi i \exp(2\pi i t)\, dt = 1$$

Its winding number about a point near ∞ is zero. Namely, for sufficiently large $|z|$

$$C_\Gamma 1(0) = \frac{1}{2\pi i}\int_0^1 \frac{1}{\exp(2\pi i t) - z} 2\pi i \exp(2\pi i t)\, dt \le C\frac{1}{|z|-1} \le \frac{1}{2}$$

More generally, the winding number of any curve at a point near infinity is zero.

Theorem 55. *[Jordan curve] If $\gamma' \in \mathcal{S}^\Delta$, The winding number takes exactly two values as z ranges over the complement of Γ.*

Proof. Let $\gamma' \in \mathcal{S}_k^\Delta$ and consider $I \in \mathcal{D}_k$. Let $t = c(I)$ and consider $z = \gamma(t) + id\gamma'(t)$ for small real numbers d. As $id\gamma'(t)$ is perpendicular to $\gamma'(t)$, z is not on the line segment $\gamma(I)$. By the chord arc condition, $\gamma(s)$ for $s \notin I$ has distance at least $c2^{k-1}$ from $\gamma(t)$ and hence for d small enough z has distance $c2^{k-2}$ from Γ. For d small enough we obtain

$$|C_\Gamma 1(\gamma(t) + id\gamma'(t)) - C_\Gamma 1(\gamma(t) + id\gamma'(t))|$$

$$\ge |\frac{1}{2\pi i}\int_{-2^{k-1}}^{2^{k-1}} \frac{1}{(s - id)\gamma'(t)} - \frac{1}{(s+id)\gamma'(t)}\gamma'(t) ds| - \epsilon$$

$$\ge |\frac{1}{2\pi i}\int_{-2^{k-1}}^{2^{k-1}} \frac{2id}{s^2 + d^2} ds| - \epsilon$$

$$\ge |\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{2i}{s^2+1} ds| - 2\epsilon = 1 - 2\epsilon$$

Similarly one proves an upper bound, thus the winding number, which is continuous and integer valued and thus constant for small positive d and similarly it is constant for small negative d, and it jumps by 1 as small d changes sign.

We define for $\epsilon < c2^k$ with small constant c

$$\alpha(\epsilon, s) = -i\int_{s-\epsilon}^{s+\epsilon} \frac{\gamma'(t)}{|\gamma'(t)|}\, dt.$$

The curves $\alpha(\epsilon,)$ for small ϵ stay at distance ϵ from the curve. (Exercise). In particular, if t is the common boundary point of some adjacent $I, I' \in \mathcal{D}_k$, then

$$\alpha(\epsilon, s) = i\epsilon(\frac{\gamma'(t)}{|\gamma'(t)|} + \frac{\gamma'(t')}{|\gamma'(t')|}).$$

Thus $\alpha(\epsilon, s)$ lies on the angular bisector of $\gamma(I)$ and $\gamma(I')$ and does not intersect these line segments.

On both curves the winding number is constant by continuity, and by the previous argument the values differ by 1.

As argued above in the case of the circle, the winding number is constant equal to zero outside a large ball containing the curve. Given any point z in the complement of the curve, consider a half line emanating from the point. If it does not reach the curve, the winding number of z equals that near infinity and is zero. If the ray does intersect the curve, it first intersects one of the curves $\gamma(t) \pm \alpha(\epsilon, s)$ for sufficiently small ϵ (Exercise). The winding number on z then coincides with the winding number on that curve. Finally, consider any line segment from a point near infinity to a point on the curve. Before reaching the curve, it must reach one of the curves $\gamma(t) \pm \alpha(\epsilon, s)$, and thus the winding number near infinity coincides with the winding number on that curve. This proves the theorem.

□

Theorem 56. *Let γ be a curve with $\gamma' \in \mathcal{S}_k^\Delta$. Consider a polynomial*

$$P(z) = \sum_{n=0}^{N} a_n z^n.$$

Then for every point z with winding number 1 *we have*

$$C_\Gamma P(z) = P(z),$$

While for every point z with winding number 0 *we have*

$$C_\Gamma P(z) = 0$$

Proof. By translation invariance, it suffices to consider $z = 0$. By subtracting a constant from the polynomial and using the information about the winding number at zero, it suffices to prove $C_\Gamma P(0) = 0$ assuming $a_n = 0$. But then the function $\frac{P(z)}{z}$ is the complex derivative of

$$Q(z) = \sum_{n=1}^{N} n^{-1} a_n z^n$$

The portion of the Cauchy integral over a line segment $I \in \mathcal{D}_k$ is then

$$\frac{1}{2\pi i} \int_I \frac{P(\gamma(t))}{\gamma(t)} \gamma'(t) dt = Q(\gamma(r(I))) - Q(\gamma(l(I)))$$

Summing over all line segments and telescoping gives $C_\Gamma P(0) = 0$. This proves the theorem. □

By approximation, the statement of the theorem continues to hold for Taylor series

$$\sum_{n=0}^{\infty} A_n z^n$$

which converge uniformly on Γ.

Theorem 57. *Let γ be a curve with $\gamma' \in \mathcal{S}_k^{\Delta}$ and assume* 0 *has winding number* 1*. Consider a Laurent polynomial*

$$P(z) = \sum_{n=-\infty}^{-1} a_n z^n.$$

that converges uniformly on Γ. Then for every point z with winding number 1 *we have*

$$C_\Gamma P(z) = 0$$

While for every point z with winding number 0 *we have*

$$C_\Gamma P(z) = -P(z)$$

Proof. By linearity and approximation, it suffices to prove the theorem for a monomial $P(z) = z^{-n}$ The claim $C_\Gamma(0) = 0$ follows as in the previous theorem by providing an explicit primitive of P/z.
Next we observe that a monomial $(z - z_0)^{-n}$ can be approximated by a polynomial at Laurent series with negative monomials at the point 0, uniformly on all points with distance greater than 2ϵ form 0
Namely,

$$(z - z_0)^{-n} = z^{-n}\left(1 - \frac{z_0}{z}\right)^{-n} = z^{-n}\left(\sum_{j=0}^{\infty}\left(\frac{z_0}{z}\right)^j\right)^n.$$

The geometric series converges uniformly an remains bounded for $|z| > 2\epsilon$. hecne we may truncate it suitably, and then the product of the truncations is uniformly close to the left hand side for $|z| > 2\epsilon$.
Now we can prove the statement for z in a small neighborhood of 0. We translate the situation so that z becomes zero and then approximate the translated monomial by a Laurent series at zero.
More generally, let z_0 be any point in the connected component of 0 and draw a path from 0 to z. The compact path stays at positive distance 2ϵ form the curve, so we may dray a sequence all discs of radius ϵ centered at the curve and iterate the above translation argument to conclude that a Laurent monomial at $z = 0$ can be approximated by a Laurent series at 0 that converges uniformly on Γ. The proves the theorem on the connected component of 0, that is the points where the winding number is 1.
To prove the theorem for the set where the winding number is 0, we first consider a point z_0 near infinity. We compute

$$\frac{z^{-n}}{z - z_0} = -z^{-n} z_0^{-1} \frac{1}{1 - z/z_0} = -z^{-n} z_0^{-1} \sum_{j=0}^{\infty} z^j z_0^{-j}$$

Approximating by finite subsums and using previous results, we find that the integral gives $-z_0^{-n}$. To argue for arbitrary points with winding number zero, we may move the pole at z_0 with a similar procedure as above. This proves the theorem. $\square$

For functions f that are continuous on the curve, we may apply the following theorem. In particular, for f continuous on the curve, the Cauchy integral is locally analytic in z on the complement of Γ.

Theorem 58 (Runge approximation)**.** *Given a curve γ with $\gamma' \in \mathcal{S}^\Delta$ and with winding number 1 about 0, we may uniformly approximate any continuous function on Γ by a Laurent series*

$$\sum_{n=-N}^{N} a_n z^n.$$

Proof. We only scetch the proof, wich is similar to the above procedure. We consider

$$\frac{1}{\gamma(t) - \alpha(\epsilon t)} - \frac{1}{\gamma(t) + \alpha(\epsilon t)}$$

for say t on a line segment $\gamma(I)$ with $I \in \mathcal{D}_k$ represents for sufficiently small ϵ a sharp real bumb on the line segment which is small outside the line segment. Any continuous function vanishing on the endpoints of all line segment $\gamma(I)$ with $I \in \mathcal{D}_k$ can be approximated by such bumps. To make the function vanish on the endpoints, we may first subtract functions as above with t parameterizing the endpoints of the corners. To approximate by Laurent series, we move the poles of the above functions to *zero* and ∞ by a procedure as before. $\square$

We consider the special case of the unit circle, $\gamma(t) = \exp(2\pi i t)$. Then the Cauchy integral is analytic and thus harmonic in the unit disc and we may study whether it is the harmonic extension of a measure.
Assuming $f \circ \gamma$ is in $\mathcal{S}^\Delta$, we compute the Taylor series about the origin.

$$C_\Gamma f(z) = \frac{1}{2\pi i} \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t) dt$$

$$= \frac{1}{2\pi i} \int_0^1 \frac{f(\gamma(t))}{\gamma(t)} \gamma'(t) \sum_{n=0}^{\infty} \frac{z^n}{\gamma(t)^n} dt$$

$$= \sum_{n=0}^{\infty} (\int_0^1 f(\gamma(t)) e^{-2\pi i n t} dt) z^n = \sum_{n=0}^{\infty} \widehat{f \circ \gamma}_n z^n$$

Recall the Poisson kernel

$$P(z) = 1 + \sum_{n>0} z^n + \overline{z}^n$$

and define the conjugate Poisson kernel

$$Q(z) = \sum_{n>0} z^n - \overline{z}^n$$

And note that

$$C_\Gamma f(z) = \int_0^1 f(\gamma(t))\frac{1}{2}(1 + P(ze^{-2\pi it}) + Q(ze^{-2\pi it})dt$$

The conjugate Poisson kernel is purely imaginary, it is equal to

$$\frac{z}{1-z} - \frac{\overline{z}}{1-\overline{z}} = \frac{z-\overline{z}}{|1-z|^2}$$

It has boundary values almost everywhere on the circle as

$$\frac{z}{1-z} - \frac{z^{-1}}{1-z^{-1}} = \frac{1+z}{1-z}$$

Note that for f in L^2, $C_\Gamma f$ is also the extension of a function in L^2, as one can see form the Fourier series.

11.1. **Exercise.** For real valued f in L^k with even integer k, the real part of $C_\Gamma f$ is the harmonic extension of $\frac{1}{2}(\int_0^1 f(x)dx + f)$ and the imaginary part of $C_\Gamma f$ is the harmonic extension of some measure $\frac{1}{2}Hf$ in L^k respectively. Hint: apply the mean value theorem to the real part of $(f + iHf)^k$ for even integers k.

12. Lecture: Boundeness of the Cauchy integral

We consider a one-periodic map $\gamma : \mathbb{R} \to \mathbb{C}$ which is chord-arc and Lipschitz,

$$c|t-s| \le |\gamma(t) - \gamma(s)| \le C|t-s|$$

for some $0 < c < C < \infty$ and all $t, s \in \mathbb{R}$. In particular,

$$G(I) = \gamma(r(I)) - \gamma(l(I))$$

defines a martingale that is bounded in absolute value above by C and below by c. For z outside the image Γ of the curve, the Cauchy integral is defined under these assumptions and depends continuously on $\gamma' \in L^\infty \ell^\infty$ and $f \in L^2 \ell^2$.

We now study the limit as z approaches the curve and thus put for the time being more stringent assumptions. We assume that γ is the primitive of a function $\gamma' \in \mathcal{S}_k^\Delta$.

We define for $\epsilon < \epsilon_0$

$$\alpha(\epsilon, s) = -i \int_{s-\epsilon}^{s+\epsilon} \frac{\gamma'(t)}{|\gamma'(t)|} \, dt$$

and note that for sufficiently small ϵ_0 depending on k and the Lipschitz and chord-arc constants of γ we have that $|\alpha(\epsilon, s)|$ is comparable to ϵ and $\gamma(s) + \alpha(\epsilon, s)$ has distance comparable to ϵ from the entire curve Γ (Exercise).

For $f \in \mathcal{S}_{k'}^\Delta$ with $k' \ge k$ we consider the Cauchy integral near the curve

$$C_{\pm\epsilon} f(s) = \frac{1}{2\pi i} \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - \gamma(s) \pm \alpha(\epsilon, s)} \gamma'(t) dt.$$

We split $C_{\pm\epsilon} f(s)$ for $s \in I$, $I \in \mathcal{D}_k$ as

$$\frac{1}{2\pi i} \int_{l(I)+\epsilon}^{r(I)-\epsilon} \frac{f(\gamma(t))}{\gamma(t) - \gamma(s) \pm \alpha(\epsilon, s)} \gamma'(t) dt + R \tag{10}$$

where the reminder is the integral over the complement over the noted interval and thanks to the uniform lower bound on $|\gamma(t) - \gamma(s)|$ on the complement it has a limit as ϵ tends to 0, which commutes with the integral by dominated convergence, as it is dominated by

$$C \|f\|_\infty \int_{[0,1)} \frac{1}{\max(|s-t|, \epsilon)} \, dt \le C \|f\|_\infty (1 + |\log(\operatorname{dist}(s, I^c))|).$$

On the main part in (10) we note $f(\gamma(t))$ is a constant, $\alpha(\epsilon, s)$ is $-2i\epsilon\gamma'(t)/|\gamma'(t)|$ and $\gamma(t) - \gamma(s) = (t-s)\gamma'(s)$. The main part is then equal to

$$\frac{f(\gamma(s))}{2\pi i} \int_{l(I)+\epsilon}^{r(I)-\epsilon} \frac{1}{t - s \mp 2i\epsilon/|\gamma'(s)|} \, dt$$

$$= \frac{f(\gamma(s))}{2\pi i} \int_{l(I)+\epsilon}^{r(I)-\epsilon} \left(\frac{t-s}{(t-s)^2 + 4\epsilon^2/|\gamma'(s)|^2} \mp \frac{2i\epsilon/|\gamma'(s)|}{(t-s)^2 + 4\epsilon^2/|\gamma'(s)|^2} \right) dt$$

The integral over the imaginary part of the integrand is bounded in terms of the structural constants of γ and by dominated convergence converges to

$$\mp i\int_{\mathbb{R}}\frac{1}{(t-s)^2+1}\,dt=\mp\pi i.$$

The difference in the two contours may be interpreted in terms of branches of the complex logarithm. Thanks to this difference, we see

$$\lim_{\epsilon\to 0}(C_{-\epsilon}-C_{+\epsilon})f(s)=f(s).$$

Similarly, $\lim_{\epsilon\to 0}C_{-\epsilon}+C_{+\epsilon}$ is given by the real part of the integrand above. By changing the integrand on a small symmetric neighborhood of s by an odd function whose integral vanishes, and using domianted convergence away of the neighborhood, this limit exists as $\epsilon\to 0$ and coincides with the limit

$$\lim_{\epsilon\to 0}\frac{1}{\pi i}f(\gamma(s))\int_{l(I)+\epsilon}^{r(I)-\epsilon}\frac{1_{|t-s|>\epsilon}}{(t-s)}\,dt.$$

Summing all intervals and using absolute integrability to pass to a limit near the endpoints of the intervals we obtain

$$\lim_{\epsilon\to 0}(C_{-\epsilon}+C_{+\epsilon})f(s)=\lim_{\epsilon\to 0}\frac{1}{\pi i}\int_{\epsilon\le|t-s|\le\frac{1}{2}}\frac{f(\gamma(t))}{\gamma(t)-\gamma(s)}\,\gamma'(t)dt.$$

Define the Hilbert transform H along the curve γ by

$$Hf=i\lim_{\epsilon\to 0}\frac{1}{\pi i}\int_{\epsilon\le|t-s|\le\frac{1}{2}}\frac{f(\gamma(t))}{\gamma(t)-\gamma(s)}\,\gamma'(t)dt,$$

so we have $Hf=i\lim_{\epsilon\to 0}(C_{-\epsilon}+C_{+\epsilon})f$.
We now remove the assumption that $\gamma'\in\mathcal{S}^{\Delta}$ and let γ be an arbitrary curve with Lipschitz and chord arc assumptions. Assume we can make sense of H in this setting, then we may define C_- and C_+ by $C_--C_+=\mathrm{id}$ and $i(C_-+C_+)=H$. We consider two functions $u,v\in\mathcal{S}^{\Delta}_k$ and study the expression

$$\frac{1}{\pi}\lim_{\epsilon\to 0}\int_0^1\int_{\epsilon<|t-s|<\frac{1}{2}}\frac{1}{\gamma(t)-\gamma(s)}u(\gamma(t))v(\gamma(s))\gamma'(t)\gamma'(s)\,dtds. \tag{11}$$

The integral can be defined for fixed ϵ by paraproducts for the bounded martingale γ'. Note that by periodicity of the integrand the domain is symmetric in t and s, so the whole expression is anti-symmetric in u and v. Breaking up the domain of integration into squares $I'\times J'$ of sidelength 2^k and noting that by a symmetry the diagonal terms $I'=J'$ vanish, and using absolute integrability outside the diagonal terms, we see that the limit can be commuted with the first integral sign by dominated convergence, hence the last display can be interpreted as a pairing

$$\langle Hu,v\rangle_{\gamma'}=-\langle u,Hv\rangle_{\gamma'}$$

with the pairing from Theorem 41

$$\langle u,v\rangle_{\gamma'} = \int_0^1 u(\gamma(s))v(\gamma(s))\gamma'(s)\,ds.$$

We shall use this symmetric expression to define H in general, and we have by arguments as above for $\gamma' \in \mathcal{S}_k^\Delta$

$$\langle Hf,h\rangle_{\gamma'} = \lim_{\epsilon\to 0}\langle C_{-\epsilon} + C_{+\epsilon}f,h\rangle_{\gamma'}.$$

We study (11) for the adapted Haar functions β_I of Theorem 41. Note by anti-symmetry we have $\langle H\beta_I,\beta_I\rangle_{\gamma'} = 0$ for all I and $\langle H1,1\rangle_{\gamma'} = 0$.

Theorem 59. *Under the assumptions above, for any interval $I \in \mathcal{D}$ we have*

$$\langle H1,\beta_I\rangle_{\gamma'} = 0.$$

For any further dyadic interval $J \neq I$ with $|I| \leq |J|$ we have

$$|\langle H\beta_I,\beta_J\rangle_{\gamma'}|$$

$$\leq C\left(1+\frac{\min(|c_I-c_J|,|c_I-l_J|,|c_I-r_J|)}{|I|}\right)^{-1}\left(1+\frac{|c_I-c_J|}{|J|}\right)^{-1}\left(\frac{|I|}{|J|}\right)^{1/2}$$

where c_I, r_I, l_I denote, center, right and left endpoint of I.

Proof. We note that for each small ϵ we have that

$$\int_0^1 \frac{1}{\gamma(t)-\gamma(s)\pm\alpha(\epsilon,s)}\gamma'(t)dt$$

is a winding number and constant in s for fixed sign $\pm$. Approximating general γ' by approximations in $\mathcal{S}_k^\Delta$ we see that this winding number is constant in the approximation as the approximation is close enough. Hence $H1$ is constant and by construction of β_I we have $\langle H1,\beta_I\rangle_{\gamma'} = 0$. Assume now $|I| \leq |J|$ and $I \neq J$. Then β_J is constant on I and we may add a constant c bounded by $\|\beta_J\|_\infty$ to β_J so that the sum vanishes on I. By the previous result,

$$\langle H\beta_I,\beta_J\rangle_{\gamma'} = \langle H\beta_I,\beta_J+c\rangle_{\gamma'}.$$

Taking limit as $\epsilon \to 0$ thanks to disjoint support in s and t and using again the construction of β_I, we write for the last display

$$\int_{0\leq t,s<1}|(\frac{1}{\gamma(t)-\gamma(s)}-\frac{1}{\gamma(c(I))-\gamma(s)})\beta_I(t)(\beta_J(s)+c)\gamma'(t)\gamma'(s)|\,dtds$$

$$\leq C(|I||J|)^{-1/2}\int_{\mathrm{supp}(\beta_J+c)}\int_I \frac{|I|}{|\gamma(t)-\gamma(s)||\gamma(c(I))-\gamma(s)|}\,dtds \tag{12}$$

If J has distance at least $|J|$ from I, then $c = 0$ and the support β_J is J and we estimate (12) by

$$\leq C(|I||J|)^{-1/2}\int_J\int_I \frac{|I|}{|c_I-c_J|^2}\,dtds = C(\frac{|I|}{|J|})^{1/2}\frac{|I||J|}{|c_I-c_J|^2}.$$

If the distance of J to I is no larger than $|J|$, and I is smaller than J, we let J^* be the interval of length $|J|/2$ containing I. If the distance of I to $(J^*)^c$ is at least I, we estimate (12) by

$$\le C(|I||J|)^{-1/2} \int_{J^{*c}} \int_I \frac{|I|}{|c_I - s|^2}\, dtds = C(\frac{|I|}{|J|})^{1/2} \frac{|I|}{\mathrm{dist}(I,(J^*)^c)}.$$

If the distance from I to J is no larger than $|I|$, we estimate (12) by

$$\le C(|I||J|)^{-1/2} \int_{(3I)^c} \int_I \frac{|I|}{|c(I) - s|^2}\, dtds$$

$$+C(|I||J|)^{-1/2} \int_{(3I)\setminus I} \int_I \frac{|I|}{|\gamma(t) - \gamma(s)||\gamma(c(I)) - \gamma(s)|}\, dtds$$

Both integrals are finite numbers: in particular for the second one we obtain in the first integral over t a logarithmic singularity, which then is integrable in s. By scaling, it suffices to estimate the integrals for I and interval of unit length. Keeping track of scaling factors, we obtain the bound

$$\le C(\frac{|I|}{|J|})^{1/2}$$

In all three cases, we have obtained the desired bound. □

Theorem 60. *Assume γ Lipschitz and chord-arc as above and $u, v \in \mathcal{S}_k^{\Delta}$. Then for a constant depending only on the structural constants of γ,*

$$\langle Hf, h\rangle_{\gamma'} \le C\|f\|_2\|g\|_2$$

Proof. We assume without loss of generality $\|u\|_2 = 1$ and $\|v\|_2 = 1$ and expand f and h into adapted Haar functions an obtain using $\langle H1, 1\rangle_{\gamma'} = 0$ and $\langle H1, \beta_I\rangle_{\gamma'} = 0$ and with Theorem 42

$$\langle Hu, v\rangle_{\gamma'} = \sum_{I,J\in\mathcal{D}} \langle u, \beta_I\rangle_{\gamma'} \langle H\beta_I, \beta_J\rangle_{\gamma'} \langle v, \beta_J\rangle_{\gamma'}$$

$$\le \sum_{I,J\in\mathcal{D}} |\langle u, \beta_I\rangle_{\gamma'}|^2 |\langle H\beta_I, \beta_J\rangle_{\gamma'}| + \sum_{I,J\in\mathcal{D}} |\langle v, \beta_J\rangle_{\gamma'}|^2 |\langle H\beta_I, \beta_J\rangle_{\gamma'}|$$

$$\le C \sup_{I\in\mathcal{D}} \sum_{J\in\mathcal{D}} |\langle H\beta_I, \beta_J\rangle_{\gamma'}| + \sup_{J\in\mathcal{D}} \sum_{I\in\mathcal{D}} |\langle H\beta_I, \beta_J\rangle_{\gamma'}|.$$

By symmetry it suffices to prove a bound on the first summand of the last display. Fix I, we aim to prove a bound on

$$\sum_{J\in\mathcal{D}} |\langle H\beta_I, \beta_J\rangle_{\gamma'}| = \sum_{k\le 0} \sum_{J\in\mathcal{D}_k} |\langle H\beta_I, \beta_J\rangle_{\gamma'}|.$$

We fix k and prove a bound on the sum $J \in \mathcal{D}_k$. If $2^k \ge |I|$ we have

$$\sum_{J\in\mathcal{D}_k} |\langle H\beta_I, \beta_J\rangle_{\gamma'}| \le C(|I|/|J|)^{1/2} \sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} \le C(|I|/|J|)^{1/2}.$$

If $2^k < |I|$ we have

$$\sum_{J\in\mathcal{D}_k} |\langle H\beta_I, \beta_J\rangle_{\gamma'}|$$

$$\leq C(|J|/|I|)^{1/2}\left(\sum_{n\in\mathbb{Z},|n|\leq 2|I|/|J|}\frac{1}{1+|n|}+\sum_{n\in\mathbb{Z},|n|>2|I|/|J|}\frac{|J|}{|I|}\frac{1}{1+n^2}\right)$$
$$\leq C|\log(|I|/|J|)|(|I|/|J|)^{1/2}.$$

Adding these estimates over all k gives the desired bound.

□

The operator H extends to a bounded operator on L^2. Likewise we obtain bounded operators

$$C_- = \frac{1}{2}1 - iHf$$
$$C_+ = \frac{1}{2}1 + iHf$$

Applying these to the dense class of Laurent series, one sees the relations

$$C_-^2 = C-$$
$$C_+^2 = -C_+$$
$$C_-C_+ = C_+C_- = 0$$

This leads to the relation

$$H^2 = -1$$

The maps C_- and $-C_+$ are projections onto subspaces of L^2, however they are not self adjoint relative to the usual inner product, they only inherit from H some symmetry relativ to the pairing studied above. Let E_- and E_+ be the orthogonal projections onto the same corresponding spaces. Then we observe

$$E_-C_- = C_-$$
$$C_-E = E$$

and hence

$$E_-C_-^* = E$$

Hence

$$E_-(1 + C_- - C_-^*) = E_- + C_- - E_- = C_-$$
$$E_- = C_-(1 + C_- - C_-^*)^{-1}$$

In the last line we notice that $C_- - C_-^*$ has purely imaginary spectrum and thus the operator in brackets is invertible.

13. Lecture: Grand embeddings

We consider the setting of $\mathbb{R}^d$. A dyadic cube is a cartesian product of dyadic intervals of equal length

$$Q = I_1 \times \cdots \times I_d$$

where the dyadic interval I_j is of the form

$$[2^k n_j, 2^k(n_j + 1))$$

with integers k, n_j. We write $s(Q)$ for the side-length 2^k of Q, and we write $\sigma(Q) = 2^{dk}$ for the volume of the cube. We write $c(Q)$ for the center of Q.

For a dyadic cube Q we define the tree T_Q to be the set of all dyadic cubes R such that $R \subset Q$ and we define $\sigma(T_Q) = \sigma(Q)$. There is a theory of $L^p\ell^q$ spaces analoguous to the setting on the interval $[0,1)$. There is also an analoguous theory martingale extension F of a Radon measure m on $\mathbb{R}^d$.

A continuous variant of the martingale averages is the grand average embedding. We fix an $0 < \epsilon \leq 1$ and define

$$F^*(Q) = \sigma(Q)^{-1} \sup_{f \in A_Q} |m(f)|,$$

where A_Q is the set of continuous functions f such that

$$|f(x)| \leq (1 + |\frac{x - c(Q)}{s(Q)}|)^{-d-\epsilon} \tag{13}$$

Theorem 61. *We have for* $1 < p \leq \infty$

$$L^p\ell^\infty F^* \leq C\|m\|_p$$

Proof. By splitting m first into real and imaginary parts and then into positive and negative parts, and using sub-linearity, we may assume m is nonnegative. Consider first $p = \infty$. We have for each Q some $f \in A_Q$

$$F^*(Q) \leq C\sigma(Q)^{-1}|m(f)| \leq C\sigma(Q)^{-1}\|m\|_\infty\|f\|_1 \leq C\|m\|_\infty.$$

This gives the bound $L^\infty\ell^\infty F^* \leq C\|m\|_\infty$. At $p = 1$, we prove a weak bound. Fix $\lambda > 0$ and consider the collection $\mathcal{E}'$ of trees T_Q where Q is a maximal dyadic cube with respect to set inclusion such that

$$|F(Q)| \geq \lambda\sigma(Q)$$

We say that two dyadic cubes Q, Q' are close, $Q \sim Q'$, if the cubes have same side-length and their dyadic parents are equal or adjacent in the sense that their closures are not disjoint. Let $\mathcal{E}''$ be the set of trees T_Q such that there is $Q' \in \mathcal{E}'$ with $Q' \sim Q$. As each cube has a bounded number of close cubes, depending on the dimension, we have

$$\sum_{\mathcal{E}''} \sigma(T_Q) \leq C \sum_{\mathcal{E}'} \sigma(T_Q) \leq C\lambda^{-1}\|m\|_1,$$

where the last inequality we recall from the martingale setting. It then suffices to show that for every $R \notin \bigcup_{Q\in\mathcal{E}''} T_Q$ we have $|F^*(R)| \le C\lambda$. There is a partition $\mathcal{R}$ of the space $\mathbb{R}^d$ containing R and dyadic cubes R' not in $\mathcal{E}'$ which are disjoint from R but have a close cube containing R. For this we use that all cubes close to a cube containing R cover all of $\mathbb{R}^d$. We then have for $f_R \in A_R$

$$|m(f_R)| \le \sum_{R'\in\mathcal{R}} |m(f_R 1_{R'})| \le \sum_{R'\in\mathcal{R}} \sigma(R')F(R')\|f_R 1_{R'}\|_\infty \le$$

$$C\lambda \sum_{R'\in\mathcal{R}} \int_{R'} \|f_R 1_{R'}\|_\infty \, dx \le C\lambda \sum_{R'\in\mathcal{R}} \int_{R'} (1+|x|/\sigma(R))^{-d-1}\, dx \le C\lambda\sigma(R)$$

This shows the weak type inequality. Marcinkiewicz interpolation completes the proof. □

The grand difference embedding of a Radon measure m on $\mathbb{R}^d$ is defined as

$$\Delta F^*(Q) = \sigma(Q)^{-1} \sup_{f\in D_Q} |m(f)| \tag{14}$$

where D_Q is the set of continuously differentiable functions in A_Q such that

$$\int_{\mathbb{R}^d} f(x)\, dx = 0$$

$$|\nabla f(x)| \le s(Q)^{-1}|1 + \frac{x - c(Q)}{s(Q)}|^{-d-1-\epsilon}. \tag{15}$$

Note that (15) implies (13) with a possible constant, if we assume that f tends to zero near ∞. We show this for Q the unit cube of side-length 1, then for $|x| \ge 1$ we have

$$|f(x)| \le \int_1^\infty |\nabla f(rx)\cdot x|\, dr \le C\int_{|x|}^\infty r^{-1-d-\epsilon}\, dr \le C|x|^{-d-\epsilon},$$

and for $|x| \le 1$ we have

$$|f(x)| \le \int_1^{\frac{1}{|x|}} |\nabla f(rx)\cdot x|\, dr + C \le C.$$

Theorem 62. *Let Q, R be two dyadic cubes with $s(Q) \le s(R)$ and let f_Q and f_R be in D_Q and D_R respectively. Then*

$$|\langle f_R, f_Q\rangle| \le C\sigma(Q)(\frac{s(Q)}{s(R)})^{1/(d+1+\epsilon)}(1 + \frac{|c(Q)-c(R)|}{s(R)})^{-d-\epsilon}.$$

Proof. Assume first $|c(Q) - c(R)| > 2s(R)$.
We consider a ball B of radius $cs(Q) < r < |c(Q)-c(R)|/2$ around $c(Q)$ and write

$$\int f_R(x)\overline{f_Q(x)}\, dx$$

$$= \int_B (f_R(x) - f_R(c(Q)))\overline{f_Q(x)}\, dx + \int_{B^c} (f_R(x) - f_R(c(Q)))\overline{f_Q(x)}\, dx.$$

We estimate the first integral in absolute value by the mean value theorem by

$$r \sup_{y\in B} |\nabla f_R(y)| \|f_Q\|_1$$
$$\leq Crs(R)^{-1}(\frac{|c(Q)-c(R)|}{s(R)})^{-d-1-\epsilon}\sigma(Q) = C\frac{rs(R)^{d+\epsilon}s(Q)^d}{(c(Q)-c(R))^{d+1+\epsilon}}$$

The second integral we estimate by

$$\|f_R\|_1 \sup_{y\in B^c}|f_Q(y)| + |f_R(c(Q)| \int_{B^c}|f_Q(x)|\,dx$$

$$\leq C\sigma(R)(\frac{r}{s(Q)})^{-d-\epsilon} + C(\frac{|c(Q)-c(R)|}{s(R)})^{-d-\epsilon}\int_{|y|>r}(\frac{|y|}{s(Q)})^{-d-\epsilon}\,dy$$
$$\leq C\frac{s(R)^d s(Q)^{d+\epsilon}}{r^{d+\epsilon}}$$

where we estimate the first term sharply and the second term as smaller than the first. The optimal r to put the two bounds of first and second integral into equilibrium is

$$r = \frac{1}{2}(c(Q)-c(R))(\frac{|s(Q)|}{|s(R)|})^{\epsilon/(d+1+\epsilon)}$$

which we note to satisfy the bounds required earlier. Inserting this r proves the desired bound of the theorem. In case $|c(Q)-c(R)| < 2s(R)$, we do a similar calculation with

$$r = s(R)(\frac{|s(Q)|}{|s(R)|})^{\epsilon/(d+1+\epsilon)}.$$

□

Theorem 63. *For $1 < p \leq \infty$ there si a constant C such that for all Radon measures m*

$$L^p\ell^2\Delta F^* \leq C\|m\|_p.$$

Proof. We first consider the case $p = 2$. We know that the $L^2\ell^2$ norm can equivalently be expressed by a square sum, hence it suffices to bound $\sum_R \sigma(R)|\Delta F^*(R)|^2$. We estimate this for suitable f_R by

$$C\sum_R \sigma(R)^{-1}|m(f_R)|^2 \leq C\|m\|_2 \|\sum_R \sigma(R)^{-1}\overline{m(f_R)}f_R\|_2. \tag{16}$$

The square of the second norm is bounded by

$$\sum_{Q,R}\sigma(Q)^{-1}\sigma(R)^{-1}\overline{m(f_R)}m(f_Q)\langle f_R, f_Q\rangle$$
$$\leq 2\sum_{s(Q)\leq s(R)}\sigma(Q)^{-1}\sigma(R)^{-1}|m(f_R)||m(f_Q)||\langle f_R, f_Q\rangle| \leq$$

$$\left(\sum_{s(Q)\leq s(R)}\frac{C}{\sigma(Q)}|m(f_Q)|^2|\langle f_R, f_Q\rangle|\right)^{\frac{1}{2}}\left(\sum_{s(Q)\leq s(R)}\frac{\sigma(Q)}{\sigma(R)^2}|m(f_R)|^2|\langle f_R, f_Q\rangle|\right)^{\frac{1}{2}}$$

To estimate the first factor, we insert the estimate of the previous theorem and note that for fixed Q

$$\sum_{s(Q)\le s(R)} (\frac{s(Q)}{s(R)})^{\epsilon/(d+1+\epsilon)}(1+\frac{|c(Q)-c(R)|}{s(R)})^{-d-\epsilon} \le C$$

by first summing over R of fixed size, recognizing the sum of $(1+|n|)^{-d-\epsilon}$ with n running over the integer lattice, and then summing a geometric series over the scales. To estimate the second factor, we note similarly for fixed R

$$\sum_{s(Q)\le s(R)} \frac{\sigma(Q)}{\sigma(R)}(\frac{s(Q)}{s(R)})^{\epsilon/(d+1+\epsilon)}(1+\frac{|c(Q)-c(R)|}{s(R)})^{-d-\epsilon} \le C$$

by noticing the sum over $(1+|n|)^{-d-\epsilon}$ with n running over the lattice with sidelength $s(Q)/s(R)$, which gives $\sigma(R)/\sigma(Q)$ times the value of the integer lattice, and then summing a geometric series over the scales. This estimates (16) by

$$\le C\|m\|_2 \left(\sum_Q \sigma(Q)^{-1}|m(f_Q)|^2\right)^{1/2}$$

Dividing by the square root gives the desired bound at $p=2$.
To obtain the bound at $p=\infty$, we fix a tree T_Q and cut a Radon measure $m\in L^\infty$ as $m = m1_{3Q} + m1_{(3Q)^c}$ and correspondingly

$$\Delta F^*(R) = \Delta F^*_{3Q}(R) + \Delta F^*_{(3Q)^c}(R).$$

For the first part we use the L^2 bound

$$\frac{1}{\sigma(Q)}\sum_{R\in T_Q}\sigma(R)|\Delta F^*_{3Q}(R)|^2 \le C\frac{1}{\sigma(Q)}\|m1_{3Q}\|_2^2 \le C\|m\|_\infty^2.$$

For the second part we use the bound

$$|\langle m1_{(3Q)^c}, f_R\rangle| \le \|m\|_\infty \int_{(3Q)^c} s(R)^{d+\epsilon}(x-c(R))^{-d-\epsilon}\,dx$$

$$\le C\|m\|_\infty \sigma(R)(s(R)/s(Q))^\epsilon$$

Summing over all $R\in T_Q$, first over those cubes of fixed size and then over all sizes, proves the desired bound.
We turn to the weak endpoint bound at $p=1$. We do a Calderon Zygmund decomposition of m with respect to $\mathcal{E}'$ as in the proof of Theorem 61. For each tree $T_Q\in\mathcal{E}'$ consider

$$m_Q = (m-c)1_Q$$

with constant C so that $m_Q(1)=0$. Let also $\mathcal{E}''$ as in the proof of Theorem 61. We need to show for each tree $T_{Q'}$ that

$$\sum_{R\in Q', R\notin\bigcup\mathcal{E}''}\sigma(R)|\Delta F^*_Q(R)|^2 \le C\lambda^2\sigma(Q')$$

We sum over fixed size of Q first. If $s(R)$ is less than $s(Q)$ then we note that Q has distance at least $s(Q')$ from Q' we obtain

$$\sigma(R)|\Delta F_Q^*(R)|^2 \le \sigma(R)^{-1}\|m_Q\|_1^2\|f_R\|_\infty^2$$

$$\le C\lambda^2\sigma(R)^{-1}\sigma(Q)^2(s(R)/s(Q'))^{2d+2\epsilon} \le C\lambda^2\sigma(R)(s(R)/s(Q'))^{2\epsilon}$$

Summing first over R of fixed scale and then over the scales gives the desired bound. If $s(R)$ is at least as large as $s(Q)$, we estimate after subtracting a constant from f_R and using the mean value theorem

$$\sigma(R)|\Delta F_Q^*(R)|^2 \le C\sigma(R)^{-1}\|m_Q\|_1^2 s(Q)^2 \sup_{x\in Q}|\nabla f_R|^2$$

$$\le C\sigma(R)^{-1}\lambda^2\sigma(Q)^2(1+\frac{|c(Q)-c(R)|}{s(R)})^{-2(d+1+\epsilon)}$$

Summing first over R of fixed scale and then over the scales gives the desired bound. This completes the proof of the weak type endpoint at $p=1$ and by marcinkiewicz interpolation completes the proof of the theorem. □

To see some application of the last theorem, let $\phi:\mathbb{R}^d\to\mathbb{C}$ be a function such that it's Fourier transform $\widehat{\phi}$ is supported in the annulus $1<|\xi|<2$ and is rotationally symmetric. Then

$$\int_0^\infty |\widehat{\phi}(t\xi)|^2\frac{dt}{t}$$

is rotation and dilation invariant and thus a constant outside the origin. We assume ϕ is normalized so that this constant is 1. We claim that for a Schwartz function f we have for all $z\in\mathbb{R}^d$

$$Tf(z) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^\infty f(x)t^{-d}\phi(\frac{y-x}{t})\overline{\phi(\frac{y-z}{t})}\,dydx\frac{dt}{t}$$

and thus T is the identity operator. Taking formally the Fourier transform of the last display, the double convolution turns into double multiplication and we obtain

$$\widehat{Tf}(\xi) = \int_0^\infty \widehat{f}(\xi)\widehat{\phi}(t\xi)\overline{\widehat{\phi}(t\xi)}\frac{dt}{t} = \widehat{f}(\xi),$$

proving the claim, Any operator of the form

$$Tf(z) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^\infty f(x)t^{-d}\phi(\frac{y-x}{t})\overline{\psi(\frac{y-z}{t})}\,dydx\frac{dt}{t}$$

with any functions ϕ,ψ having integral zero and satisfying the estimates

$$|\phi(x)|\le(1+|x|)^{-d-\epsilon},$$

$$|\nabla\phi(x)|\le(1+|x|)^{-d-1-\epsilon},$$

will be bounded in L^p. Namely, consider the pairing

$$\langle Tf,g\rangle = \int_{\mathbb{R}^d}\int_0^\infty\left(\int_{\mathbb{R}^d}f(x)t^{-d}\phi(\frac{y-x}{t})\,dx\right)\overline{\left(\int_{\mathbb{R}^d}g(z)t^{-d}\phi(\frac{y-z}{t})\,dz\right)}$$

The exterior double integral we may split over dyadic cubes, in the sense that we split the t integral into intervals $2^k, 2^{k+1}$ and if t is in such an interval we split the y integral over dyadic cubes of side-length 2^k. This partitions the integration domain and we may estimate the last display by

$$\sum_k \sum_{s(Q)=2^k} \sigma(Q) \sup_{2^k \le t \le 2^{k+2}} \sup_{x \in Q} \left(\int_{\mathbb{R}^d} f(x) t^{-d} \phi(\frac{y-x}{t})\, dx \right) \overline{\left(\int_{\mathbb{R}^d} g(z) t^{-d} \phi(\frac{y-z}{t})\, dz \right)}$$
$$\le C \sum_k \sum_{s(Q)=2^k} \sigma(Q) F^*(Q) \overline{G^*(Q)}$$

which can then be estimated by Hölder's inequality in the spaces $L^p(\ell^2)$ and by the embedding theorems.
A typical application is to translation and dilation invariant operators, such as the Riesz transforms

$$R_i = \partial_i \Delta^{-1/2}$$

of which the Hilbert transform on the real line is a special case. We may express $R_i f$ by the Calderón's reproducing formula, applying R_i to ϕ inside the integral. Both ϕ and ψ satisfy the requirements listed as above. Boundedness of the Riesz-transform follows.

14. The corona theorem

Let $H^\infty(D)$ in this lecture denote the set of bounded analytic functions in the unit disc. We know that each such function is harmonic and thus the harmonic extension of a Radon measure in L^∞ on $\mathbb{T}$. This measure is determined by the radial limits of the function almost everywhere on $\mathbb{T}$. The space $H^\infty(D)$ is an algebra of functions, and the corona theorem is motivated by questions on the spectrum of this algebra, which we will not elaborate here. The name corona appeals to behaviour near the boundary of the disc.

Theorem 64 (Corona Theorem)**.** *Let $n \in \mathbb{N}$ and $\delta > 0$. There is a constant $C = C(n,\delta)$ such that for all tuples*

$$f_1, \dots, f_n \in H^\infty(D)$$

such that for all z in D

$$\delta < \sum_{m=1}^n |f_m(z)|^2 < 1$$

there exist a tuple

$$g_1, \dots, g_n \in H^\infty(D)$$

with $\|g_j\|_\infty \le C$ for all j and

$$\sum_{j=1}^n f_j g_j = 1.$$

The f_j are called the corona data, the g_j are called the corona solution.

Proof. We first reduce to the case when the f_j extend analytically to a disc about the origin of radius larger than 1. Fo the reduction, let f_j be as in the theorem and define $f_{r,j}(z) = f_j(rz)$ for r smaller than 1. Assume we can find for each such corona data a solution $g_{r,j}$ as in the theorem. Pick a sequence r that tends to 1. Pick consecutively subsequences so that for the n-th subsequence the Fourier coefficients $\widehat{g_{r,j}}(n)$, which remain bounded by $C(n,\delta)$, converge. Choose the diagonal subsequence, so that all Taylor coefficients of this diagonal sequence converge. By dominated convergence applied to the Taylor series, the functions $g_{r,j}$ converge uniformly on discs of radius smaller than 1 about the origin. They converge to the limit Taylor series, which gives bounded analytic functions on the unit disc with uniform bounds by $C(n,\delta)$. By taking limits these functions solve the corona problem for the original data f_j.
Henceforth we assume the functions f_j extend to holomorphic functions on discs of radius slightly larger than 1 around the origin.
Define

$$h_j = \frac{\bar{f}_j}{\sum_{m=1}^n |f_m|^2}.$$

Then $\|h_i\|_\infty \leq \delta^{-1}$ and

$$\sum_{i=1}^n f_i h_i = 1.$$

The functions h_j satisfy the desired algebraic equation, the desired L^∞ bound, and are smooth in the unit disc, but are not analytic.
Other solutions to the algebraic equation are given by adding an anti-symmetric linear matrix applied to the vector f_j

$$g_j = h_j + \sum_{l=1}^n (w_{jl} - w_{lj}) f_l,$$

as can be seen from

$$\sum_{j=1}^n f_j \sum_{l=1}^n (w_{jl} - w_{lj}) f_l = 0.$$

Such g_j will be analytic if

$$\partial_{\bar{z}} w_{jl} = h_j \partial_{\bar{z}} h_l.$$

Namely, using analyticity of f_j, that is $\partial_{\bar{z}} f_j = 0$, we obtain

$$\partial_{\bar{z}} g_j = \partial_{\bar{z}} h_j + \sum_{l=1}^n (\partial_{\bar{z}} w_{jl} - \partial_{\bar{z}} w_{lj}) f_l$$

$$= \partial_{\bar{z}} h_j + \sum_{l=1}^n h_j (\partial_{\bar{z}} h_l) f_l - \sum_{l=1}^n h_l (\partial_{\bar{z}} h_j) f_l$$

$$= \partial_{\bar{z}} h_j + \sum_{l=1}^n h_j (\partial_{\bar{z}} h_l f_l) - \partial_{\bar{z}} h_j = 0,$$

where the second summand is seen to be zero by pulling the sum inside the differentiation.
Fix j and l and set $u = h_j \partial_{\bar{z}} h_l$. We seek to find a solution to

$$\partial_{\bar{z}} w = u$$

with $\|w\|_\infty \leq C(n, \delta)$. This will complete the construction of the Corona solution.
The above equation is called the d-bar equation, it is closely related to the more widely known Poisson equation

$$\Delta \phi = u.$$

Namely, given a solution ϕ to the Poisson equation, $w = 4\partial_z \phi$ solves the d-bar equation.
Recall Green's identity

$$\int_U \phi \Delta \psi - \psi \Delta \phi \, dxdy = \int_{\partial U} \phi (\nabla \psi \cdot n) - \psi (\nabla \phi \cdot n) dS.$$

In the case of a disc D_r of radius r about the origin with infinitesimally small punctured hole at the origin this gives

$$u(0) = C \int_{D_r} \Delta u(z) |log|z/r|| \, dxdx + \int_0^1 u(re^{2\pi i\theta}) \, d\theta$$

for some constant C. With a right hand side of Poisson's equation supported on some disc of radius $1+\epsilon$ we may solve the Poisson equation for z in the support of the right-hand-side by

$$\phi(z) = C \iint_{\mathbb{R}^2} u(\zeta) \log|z-\zeta| \, d\xi d\eta$$

we assume this to be known. A solution to the d-bar equation is then given by differentiation under the integral sign, so with a possibly different constant,

$$w(z) = C_0 \iint_{\mathbb{R}^2} u(\zeta) \frac{1}{z-\zeta} \, d\xi d\eta + C_1$$

Where we added a constant so that we may assume $w(0) = 0$.
While we know that w is bounded on $\mathbb{T}$ because it extends smoothly across $\mathbb{T}$, we do not know that the bound is by a constant depending only on n and δ.
However, we may prove a slightly weaker bound. Let $H^1(D)$ be the complex Hardy space of analytic functions in the unit disc which are harmonic extensions of absolutely continuous Radon measures. We claim that for every $v \in H^1(D)$ we have

$$|\int_0^{2\pi} w(e^{2\pi i\theta}) v(e^{2\pi i\theta}) \, d\theta| \le C(n,\delta) \|v\|_1 \tag{17}$$

where the integral over the boundary is in the sense of a limit of integrals over smaller radii. This is almost the desired $L\infty$ bound: if we had this estimate for all harmonic extensions of absolutely continuous Radon measure, $v \in L^1(\mathbb{T})$, then we had the desired L^∞ bound by the converse to Hölder's inequality.
By approximation of an arbitrary $v \in H^1(D)$ by $v_r(z) = v(rz)$, we may again assume that v extends to a disc of radius slightly larger than 1. Namely, bounds by a constant $C(n,\delta)$ on the approximations turn into the bounds in the limit $r \to 1$, as we know that the limit exists as w is in L^∞. We may also assume that the approximations do not have zeros on $\mathbb{T}$ by avoiding the countably many critical radii of the discrete zeros of v.
Applying Green's formula for fixed v of norm 1 we obtain for the left hand side of (17) up to a constant

$$\iint_D \Delta(vw) |\log|z|| \, dxdy.$$

Note that

$$\partial_z \partial_{\bar{z}}(vw) = 4\partial_z(vu) = 4\partial_z(vh_j \partial_{\bar{z}} h_l)$$
$$= 4(\partial_z v) h_j \partial_{\bar{z}} h_l + 4v(\partial_z h_j)\partial_{\bar{z}} h_l + 4v h_j (\partial_z \partial_{\bar{z}} h_l). \tag{18}$$

We estimate the three terms separately. With $a = \sum |f_m|^2$ and $h_j = a^{-1}\overline{f_j}$ we have for the first term in (18)

$$4(\partial_z v) h_j a^{-1} \overline{\partial_z f_l} - 4 \sum_m (\partial_z v) h_j a^{-2} \overline{f_j} f_m \overline{\partial_z f_m} \tag{19}$$

We again estimate the terms separately. In the first term we estimate h_j and a^{-1} by their L^∞ bounds and it remains to estimate

$$\iint_D (\partial_z v)(\overline{\partial_z f_l})|\log|z||\,dxdy. \tag{20}$$

Now we consider for each $I \in \mathcal{D}$ the region I^* defined to consist of those $z = re^{2\pi i\theta}$ such that $\theta \in I$ and $1 - |I| \le r < 1 - |I|/2$. For a function f on the disc, define the embedded function

$$f^*(I) = \sup_{z \in I^*} |f(z)|$$

Integrating the logarithm over each of the regions, we estimate (20) by

$$C\sum_{\mathcal{D}} (\partial_z v)^*(I)(\overline{\partial_z f_l})^*(I)|I| \le L^1\ell^2(\partial_z v)^* L^\infty \ell^2(\overline{\partial_z f_l})^*.$$

The second term we bound by the grand embedding theorem, writing with the Cauchy integral for the derivative

$$\partial_z f_l(z) = -\int_0^1 \frac{f_l(e^{2\pi i\theta})}{(z - e^{2\pi i\theta})^2} e^{2\pi i\theta}\,d\theta$$

and noticing that the kernel satisfies the assumptions of the grand embedding theorem on the disc relative to I if $z \in I^*$.

To see that $(\partial_z v)^*$ satisfies good $L^1\ell^2$ bounds, we cannot apply the grand embedding theorem directly as this theorem does not apply for $p = 1$. The function v is analytic in a neighborhood of the closed unit disc and does not vanish on the boudnary of the unit disc. It has finitely many zeros $z_1, \dots, z_N$ in the unit disc, here we list a zero with repetition according to its multiplicity. Note that the socalled Blaschke factor

$$B_j(z) = \frac{z - z_j}{1 - z\overline{z_j}}$$

vanishes at z_j an takes modulus one on the boundary of the disc, as there it equals

$$\frac{z - z_j}{z(\overline{z - z_j})}.$$

Hence

$$v\prod_{j=1}^N B_j(z)^{-1}$$

is analytic in the unit disc, has no zeroes there, and has the same modulus as v on the boundary of the unit disc. Thanks to non-vanishing, we may take an analytic square root s of the function on the simply connected disc. This square root is then in L^2, as its square is integrable. We then have

$$v = s^2 \prod_{j=1}^N B_j(z)$$

$$\partial_z v = 2(\partial_z s)s\prod_{j=1}^N B_j(z) + s^2\partial_z(\prod_{j=1}^N B_j(z))$$

Now s is in $H^2(D)$, and by the embedding theorems as above $(\partial_z s)^*$ is in $L^2\ell^2$, s^* is in $L^2\ell^\infty$ by the grand embedding applied to the representation of s by the Poisson kernel, and the product $(\prod_{j=1}^N B_j(z))^*$ is in $L^\infty\ell^\infty$ and $(\partial_z(\prod_{j=1}^N B_j(z)))^*$ is in $L^\infty\ell^2$. By Hölder, v^* is in $L^1(\ell^2)$ and thus in the discrete Hardy space, with norm controlled by the complex Hardy space norm of v. This proves the desired bound on the first term in (19).

The second term in (19) is estimated similarly, using L^∞ bounds for f_j.

The second term in (18) is written as

$$s^2\prod_{j=1}^N B_j(z)(\partial_z h_j)\partial_{\bar z} h_l$$

we then use that the Blaschke factors are bounded, that $L^2\ell^\infty s^*$ is bounded by the grand embedding theorem applied to the representation of s as Poisson extension, and that $L^\infty\ell^2\partial_z h_j$ and $L^\infty\ell^2\partial_{\bar z} h_j$ are bounded.

Similarly we estimate the third term in (18), where we use

$$\partial_z\partial_{\bar z} h_l = \partial_z(a^{-1}\partial_{\bar z}\overline{f_l} - \sum_m a^{-2}\overline{f_l} f_m \partial_{\bar z}\overline{f_m})$$

$$= \sum_m a^{-2}(\partial f_m)\overline{f_m}\partial_{\bar z}\overline{f_l} - \sum_m a^{-2}\overline{f_l}(\partial_z f_m)\partial_{\bar z}\overline{f_m} + 2\sum_m\sum_{m'} a^{-3}\overline{f_l}(\partial_z f_{m'})\overline{f_{m'}} f_m \partial_{\bar z}\overline{f_m}$$

Similarly to before we estimate the derivative terms using $L^\infty\ell^2$, we estimate s by $L^2\ell^\infty$ and all other terms are bounded. This completes the proof of (17)

Now (17) defines a bounded linear functional on $H^1(D)$. We apply the Hahn Banach theorem to extend this functional to a functional on $L^1(\mathbb{T})$, with the same bound. Here we use that $H^1(D)$ is a closed subspace of $L^1(T)$. By the Riesz representation theorem, this functional is given by an element $w' \in L^\infty(\mathbb{T})$, which satisfies the bound $\|w'\|_\infty \le C(n,\delta)$.

We claim that w' also solves the d-bar equation. For this we need to show that $w - w'$ is analytic in the unit disc.

We have $w - w' \in L^\infty(\mathbb{T})$, so it has a harmonic extension to the unit disc. We have

$$\int_0^{2\pi}(w - w')e^{2\pi i n\theta}\,d\theta = 0$$

for all $n \ge 0$ as $e^{2\pi i n\theta}$ is the boundary value of the function $z^n \in H^1(D)$ and w and w' induce the same functional on $H^1(D)$. Hence the Fourier series of the harmonic extension of $w - w'$ has only nonnegative frequencies and thus is a Taylor series and thus the harmonic extension is analytic.

This completes the proof of the theorem.

□