HARMONIC ANALYSIS SUMMER 2020

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1. Lecture: Harmonic functions on the disc, I

A classical theme in harmonic analysis concerns harmonic functions on the open unit disc

$$\mathbb{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

We may identitfy the plane with the complex plane and write

$$x + iy = z.$$

We write for a function on the disc $u(z) = u(x,y)$, where one argument stands for a complex number and two arguments for a pair of real numbers.
A complex valued function $f : D \to \mathbb{C}$ is called harmonic, if it is twice continuously differentiable and

$$\Delta u = 0,$$

where

$$\Delta u = \partial_x^2 u + \partial_y^2 u = 4\partial_z \partial_{\overline{z}} u$$

with

$$\partial_z u = \frac{1}{2}(\partial_x u - i\partial_y u)$$
$$\partial_{\overline{z}} u = \frac{1}{2}(\partial_x u + i\partial_y u).$$

For each n, both of the functions

$$u(z) = z^n$$
$$u(z) = \overline{z}^n$$

are harmonic since $\partial_{\overline{z}} z^n = 0$ and conjugation preserves harmonic functions.
In the unit disc, it is natural to pass to polar coordinates. A continuous function u on the unit disc becomes a continuous function $\tilde{u}$ on the half plane $\mathbb{R}_{\geq 0} \times \mathbb{R}$

$$\tilde{u}(r,\theta) = u(r\cos(2\pi\theta), r\sin(2\pi\theta))$$

which is 1 -periodic in θ and constant on the line $r = 0$. If u is twice continuously differentiable, we have that $\tilde{u}$ is twice continuously differentiable on $r > 0$. We have

$$\Delta u(r\cos(2\pi\theta), r\sin(2\pi\theta)) \tag{1}$$

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$$= \partial_r^2 \tilde{u}(r,\theta) + r^{-1}\partial_r \tilde{u}(r,\theta) + (2\pi r)^{-2}\partial_\theta^2 \tilde{u}(r,\theta)$$

and this expression has a continuous extension to $r = 0$ which is constant on $r = 0$. We omit the tilde on the u if it is clear form the context that we use polar coordinates.
Consider the circular averages

$$a(r) = \int_0^1 u(r,\theta)\,d\theta.$$

Lemma 0 (Mean value principle). *If u is harmonic on the open unit disc, then its circular average a is constant in r and equal to $u(0)$.*

Proof. The function a is twice continuously differentiable for $r > 0$ and we have

$$\partial_r(r\partial_r a) = (\partial_r + r\partial_r^2)a$$
$$\int_0^1 (\partial_r + r\partial_r^2)u(r,\theta)\,d\theta$$
$$= -\int_0^1 (2\pi)^{-2} r^{-1} \partial_\theta^2 r u(r,\theta)\,d\theta = 0.$$

The last by periodicity of u in θ. Hence

$$r\partial_r a = c_1,$$
$$a = c_1 \ln r + c_2.$$

If u remains bounded in the disc, then necessarily c_1 vanishes and a is constant. To determine the constant, we let r tend to 0. Since $u(r,\theta)$ tends to $u(0)$ as $r \to 0$ uniformly in θ, we have $\lim_{r\to 0} a(r) = u(0)$, and we have proven the lemma.

□

Lemma 1. *If a real valued harmonic function in the open unit disc attains a maximum, then it is constant.*

Proof. Assume first it attains its maximum at the origin. Fix $0 < r < 1$. We have by pointwise estimation of the integrand

$$\int_0^1 u(r,\theta)\,d\theta \le \int_0^1 u(0)\,d\theta$$

with equality if and only if we have equality at every point of the integrand. Here we have used that both integrands are continuous and for continuous functions the integral is strictly monotone in the integrand. However, the left hand side is $u(0)$ by the mean value property and the right hand side is $u(0)$ by direct calculation. So we have for every θ

$$u(0) = u(r,\theta).$$

Since r was arbitrary, this shows that u is constant. Now assume u attains a maximum at some point possibly other than the origin. By a compactness argument there is a point z of minimal absolute value where the maximum is attained. Assume to get a contradiction that z is not zero. We consider a small disc contained in the unit disc centered at z. By rescaling the small disc to the unit disc and the harmonic function on the small disc to a harmonic function on the unit disc, we obtain from the rpevious argument that the function is constant on the small open disc. The function therefore attains its maximum on a point with smaller absolute value than z, a contradiction. So the function attains its maximum at the origin. As we have seen, it is constant. $\square$

Denote the boundary of $\mathbb{D}$, the unit circle, by

$$\mathbb{T} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

In polar coordinates, functions on the circle become 1-periodic functions in the angular variable θ. The Wiener algebra on $\mathbb{T}$ is the set of all functions of the form

$$f(\theta) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n\theta} \tag{2}$$

with a sequence of complex numbers

$$(c_n)_{n\in\mathbb{Z}} \in l^1(\mathbb{Z}),$$

that is

$$\sum_{n\in\mathbb{Z}} |c_n| < \infty.$$

The series in (2) is called a Fourier series. Note that an element in the Wiener algebra is a continuous function by uniform convergence of this Fourier series. However, it is well known that not every continuous function can be written in a Fourier series with absolutely summable coefficients.

The coefficients c_n are uniquely determined by f. Namely, computing the so-called Fourier coefficients $\widehat{f}(n)$ of f we obtain

$$\widehat{f}(n) := \int_0^1 f(\theta)e^{-2\pi i\theta}\, d\theta = \sum_{m\in\mathbb{Z}} c_m \int_0^1 e^{2\pi i m\theta} e^{-2\pi i n\theta}\, d\theta = c_n.$$

Here we have commuted integral and summation by uniform convergence of the Fourier series, and we have used that the last integral is nonzero only if $n = m$, in which case it is 1. Hence c_n can be recovered from f.

The functions

$$u(r, \theta) = r^{|n|} e^{2\pi i n\theta} \tag{3}$$

are harmonic, because they coincide with z^n for positive n and with $\overline{z}^{-n}$ for negative n.

Given f in the Wiener algebra, consider on $\mathbb{D} \cup \mathbb{T}$

$$u(r, \theta) = \sum_n \widehat{f}(n) r^{|n|} e^{2\pi i n\theta}. \tag{4}$$

By uniform convergence, the series is a continuous function on the closed disc $\mathbb{D} \cup \mathbb{T}$ and coincides with f on the boundary $\mathbb{T}$. By uniform

convergence on compact subdiscs of the series and its second derivative of $\mathbb{D}$, the function is harmonic in the open unit disc $\mathbb{D}$. We call u as in (3) the harmonic extension of f on the unit disc.
We now obtain a different representation of this harmonic extension. With the definition of the Fourier coefficients we have

$$u(r,\theta) = \sum_n \Big(\int_0^1 f(\phi)e^{-2\pi i\phi}\, d\phi\Big) r^{|n|} e^{2\pi in\theta}$$

$$= \int_0^1 f(\phi) \sum_n e^{2\pi i(\theta-\phi)} r^{|n|}\, d\phi = \int_0^1 f(\phi) P(r,\theta-\phi)\, d\phi$$

with the Poisson kernel

$$P(r,\theta) = \sum_n e^{2\pi i(\theta)} r^{|n|} = 1 + \sum_{n>0} z^n + \overline{z}^n$$

$$= Re\Big(1 + \frac{2z}{1-z}\Big) = Re\Big(\frac{1+z}{1-z}\Big) = \frac{1}{2}\Big(\frac{1+z}{1-z} + \frac{1+\overline{z}}{1-\overline{z}}\Big)$$

$$= \frac{1-z\overline{z}}{(1-z)(1-\overline{z})} = \frac{1-|z|^2}{|1-z|^2}.$$

Note that the Poisson kernel is positive in the open unit disc, as evident from the last expression since $|z| < 1$. It is moreover continuous in the closed unit disc without the point $z = 1$ and vanishes on the boundary without the point $z = 1$. For real $z = r$ we have

$$P(r,0) = \frac{1-r^2}{(1-r)^2} = \frac{1+r}{1-r},$$

which tends to ∞ as r tends to 1.
The Poisson kernel is harmonic in the open unit disc, as the series and its second derivative converge uniformly on compact subsets. By the mean value theorem we have

$$\int_0^1 P(r,\theta)\, d\theta = P(0) = 1.$$

For any positive ϵ, the Poisson kernel is uniformly continuous on $\mathbb{D}$ without the open ball of radius ϵ about the point 1, hence

$$\lim_{r\to 1} \int_\epsilon^{1-\epsilon} P(r,\theta)\, d\theta = \int_\epsilon^{1-\epsilon} \lim_{r\to 1} P(r,\theta)\, d\theta = 0.$$

In other words, there is an $r_\epsilon < 1$ such that for $r_\epsilon < r < 1$ we have

$$0 < \int_\epsilon^{1-\epsilon} P(r,\theta)\, d\theta \le \epsilon.$$

The right hand side of the above formula

$$u(r,\theta) = \int_0^1 f(\phi) P(r,\theta-\phi)\, d\phi \tag{5}$$

makes sense for continuous functions. This is used in the following lemma.

Lemma 2. *Let f be continuous on $\mathbb{T}$. Then there is a continuous function u on the closed disc which coincides with f on the boundary of the disc and is harmonic in the open unit disc.*

Proof. We define u in the open disc by formula (5). Since the Poisson kernel is harmonic in the open disc, so is u, as we can see from pulling the Laplace operator inside the integral. It remains to show that u extends continuously to the boundary and coincides with f there. We have to show that for all τ and all $\eta > 0$ there is an ϵ such that for $1 - \epsilon < r < 1$ and $|\theta - \tau| < \epsilon$ we have

$$|u(r, \theta) - f(\tau)| \leq \eta.$$

Since f is continuous, there is C such that

$$\|f\|_\infty := \sup_\theta |f(\theta)| \leq C \tag{6}$$

and there is an $\epsilon > 0$ such that for $|\phi - \tau| \leq 2\epsilon$ we have

$$|f(\phi) - f(\tau)| \leq \eta/2$$

and we have for $1 - \epsilon < r < 1$ and $\epsilon < |\phi| < 1 - \epsilon$ that

$$P(r, \phi) \leq \eta/(4C).$$

Then for for $1 - \epsilon < r < 1$ and $|\theta - \tau| < \epsilon$ we have

$$\begin{aligned}
|u(r,\theta) - f(\tau)| &= |\int_0^1 (f(\phi) - f(\tau))P(r, \theta - \phi)\, d\phi| \\
&= |\int_0^1 (f(\theta - \phi) - f(\tau))P(r, \phi)\, d\phi| \\
&\leq \int_{-\epsilon}^{\epsilon} (\eta/2)P(r, \phi)\, d\phi + \int_{\epsilon}^{1-\epsilon} (2C)P(r, \phi) \\
&\leq \eta/2 + 2C(\eta/4C) = \eta.
\end{aligned}$$

This completes the proof. □

The following is a uniqueness result for the harmonic extension.

Lemma 3. *Consider two functions that are continuous on the closed unit disc, harmonic on the open unit disc, and coincide on the boundary of the disc. Then they are equal on the entire closed disc.*

Proof. Assume we have two harmonic functions u and v on the open disc with continuous extensions to the closed disc which coincide on the boundary. Then $u - v$ is harmonic on the open disc and continuous on the closed disc and vanishes on the unit circle. Its real part therefore has a maximum on the closed disc, and this maximum is 0 or else it is larger than zero and attained at an interior point, and by the maximum principle $u - v$ is constant, a contradiction to its boundary values being zero. Arguing similarly for $v - u$ and $i(u - v)$ and $-i(u - v)$ we see that $u - v$ is zero. □

1.1. **Exercises.**

1.1.1. *Laplace inpolar coordinates.* Prove formula (1).

1.1.2. *Non-integrable Fourier series of continuous function.* Compute the Fourier series of the one-periodic function χ_a which is 1 on the interval $[0, a)$ and zero on the interval $[a, 1)$ for some $0 < a < 1$. Show that the Fourier coefficients of are not absolutely summable.
Prove upper and lower bounds on the Fourier coefficients for small a of the continuous function f whose derivative is

$$f'(x) = \chi_a(x) - \chi_a(x - 0.5)$$

outside the points of discontinuity of the right hand side.
Take a series of multiples of functions as above to show there is a continuous functions whose Fourier series is not summable.

2. Lecture: Harmonic functions on the unit disc, II

For functions in the Wiener algebra, we have two expressions for the harmonic extension, one by Fourier series and one with the Poisson integral.
For a continuous function f, we defined the harmonic extension by the Poisson kernel formula.

$$u(r,\theta) = \int_0^1 f(\phi)P(r,\theta-\phi)\,d\phi$$
$$= \int_0^1 f(\phi)\sum_{n\in\mathbb{Z}} r^{|n|}e^{2\pi in(\theta-\phi)}\,d\phi$$

While a continuous function cannot necessarily be written as an absolutely convergemnt Fourier series, the Fourier coefficients are well defined and one can use them to write the harmonic extension.
For $r<1$ we can inter change sum and integration in the previous display to equate the above with

$$= \sum_{n\in\mathbb{Z}} r^{|n|}e^{2\pi in\theta}\left(\int_0^1 f(\phi)e^{-2\pi in\phi}\,d\phi\right)$$
$$= \sum_{n\in\mathbb{Z}} r^{|n|}e^{2\pi in\theta}\widehat{f}(n)$$

The last sum is absolutely summable for $r<1$, but for $r=1$ it is only absolutely summable if the continuous function is in the Wiener algebra.
The following lemma characterizes those harmonic extensions of continuous functions, which come form a function in the Wiener algebra.
The Wiener algebra norm of a function f the quantity

$$\|f\|_{A^1} = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|.$$

Lemma 4. *Let f be in the Wiener algebra on the circle $\mathbb{T}$ and let u be its harmonic extension to the unit disc. Then we have for every $r<1$*

$$\|u(r,.)\|_{A^1} \le \|f\|_{A^1}. \tag{7}$$

Conversely, let u be a harmonic function in the open unit disc. It coincides with the harmonic extension of a function in the Wiener algebra if there is a constant C such that for all $r<1$

$$\|u(r,.)\|_{A^1} \le C.$$

Proof. For the first part, consider the harmonic extension u of f and observe that the Fourier coefficients of $u(r,.)$ are by (4)

$$r^{|n|}\widehat{f}(n).$$

Therefore

$$\|u(r,.)\|_{A^1} = \sum_{n\in\mathbb{Z}} r^{|n|}|\widehat{f}(n)| \le \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|.$$

This proves (7).

Now consider a harmonic function u in the open disc. Define for $0 < r < 1$ the function
$$f_r(\theta) = u(r, \theta)$$
in the Wiener algebra. Let $r < s$ and note that we have by applying the previous computation to the scaled disc of radius s
$$\widehat{f_r}(n) = \widehat{f_s}(n)(r/s)^{|n|}.$$
Define
$$c_n = r^{-|n|}\widehat{f_r}(n),$$
which by the previous display is independent of $0 < r < 1$. We have
$$\sum_n |c_n| r^{|n|} \le C$$
for all $0 < r < 1$. By monotone convergence, as $r \to 1$, we have
$$\sum_n |c_n| \le C.$$
Define the function
$$f(\theta) := \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n\theta},$$
which by the previous display is in the Wiener algebra. Then the harmonic extension of f coincides with u in the open unit disc. □

Having solved the Dirichlet problem for continuous boundary data, we turn to more general boundary data.
Let $C(\mathbb{T})$ be the linear space of continuous complex valued functions on the circle $\mathbb{T}$. A Radon measure on the circle is a linear map
$$m : C(\mathbb{T}) \to \mathbb{C}$$
satisfying
$$|m(f)| \le C\|f\|_\infty$$
for some positive constant C and where $\|f\|_\infty$ was defined in (6). We define the total mass of m to be
$$\|m\|_M = \sup_{\|f\|_\infty \le 1} |m(f)|.$$
Given a continuous function g in $\mathbb{T}$, the map
$$f \to \int_0^1 f(\theta)g(\theta)\, d\theta$$
is an example of a Radon measure. We have
$$|\int_0^1 f(\theta)g(\theta)\, d\theta| \le \|f\|_\infty \|g\|_1$$
where
$$\|g\|_1 = \int |g(\theta)|\, d\theta.$$
Another example of a Radon measure is the map
$$f \to f(0)$$

We have

$$|f(0)| \le \|f\|_\infty$$

Let m be a Radon measure on $\mathbb{T}$. Then we can define the Fourier coefficients

$$\widehat{m}(n) = m(e^{-2\pi i n .}).$$

As we have $\|e^{-2\pi i n .}\|_\infty = 1$, the Fourier series of a measure satsifies the bound:

$$|\widehat{m}(n)| \le \|m\|_{M^1}.$$

Theorem 5. *Let m be a Radon measure on $\mathbb{T}$. The function*

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} \widehat{m}(n) r^{|n|} e^{2\pi i n\theta} \tag{8}$$

is harmonic in the unit disc and satisfies for all $r < 1$

$$\int_0^1 |u(r,\theta)|\, d\theta = \|u(r,.)\|_1 \le \|m\|_{M^1}.$$

We call the function u in this theorem the harmonic extension of m in the disc. In case the measure is integration against a continuous function g, this definition coincides with the definition of the harmonic extension of g.

Proof. As the Fourier coefficients of a measure are bounded, the series (8) converges together with its second derivative uniformly on compact subsets in the open unit disc. Since all summands are harmonic, the series is a harmonic function.

We approximate the signum function of $u(r,.)$ by a continuous function h that is bounded in absolute value by 1,

$$h(\theta) = \frac{|u(r,\theta)|}{u(r,\theta)} \left(\frac{|u(r,\theta)|}{\|u(r,.)\|_\infty} \right)^\epsilon$$

with $\epsilon > 0$ close to 0. It suffices to prove for all ϵ,

$$\int_0^1 u(r,\theta) h(\theta)\, d\theta \le \|m\|_{M^1}.$$

Approximating the left hand side by Riemann integrals, it suffices to show

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N u(r, k/N) h(k/N) \le \|m\|_{M^1}.$$

With the definition of u, the left hand side becomes

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N \sum_{n\in\mathbb{Z}} m(e^{-2\pi i n .}) r^{|n|} e^{2\pi i n k/N} h(k/N)$$

Using linearity and uniform convergence we can interchange the sum and evaluation of m

$$= \lim_{N\to\infty} m\left(\frac{1}{N} \sum_{k=1}^N \sum_{n\in\mathbb{Z}} r^{|n|} e^{2\pi i n (k/N - .)} h(k/N) \right)$$

Using uniform convergence of the Riemann sums we can write this as

$$= m\left(\int_0^1 \sum_{n\in\mathbb{Z}} r^{|n|} e^{2\pi i n(\theta-.)} h(\theta)d\theta\right) = m\left(\int_0^1 P(r,\theta-.)h(\theta)d\theta\right)$$

with the Poisson kernel of the previous section. As

$$\left|\int_0^1 P(r,\theta-.)h(\theta)d\theta\right|$$

$$\leq \int_0^1 |P(r,\theta-.)|d\theta = 1,$$

we obtain

$$m\left(\int_0^1 P(r,\theta-.)h(\theta)d\theta\right) \leq \|m\|_{M^1},$$

which proves the theorem.

□

Theorem 6. *Consider a harmonic function u in the open unit disc such that for all r*

$$\int_0^1 |u(r,\theta)|\, d\theta \leq C.$$

Define for $0 \leq r < 1$ the measure

$$m_r(f) = \int_0^1 u(r,\theta)f(\theta)\, d\theta$$

Then for every $f \in C(\mathbb{T})$ the limit

$$m(f) = \lim_{r\to 1} m_r(f)$$

exists and defines a Radon measure m on $\mathbb{T}$ with $\|m\|_{M^1} \leq C$.

We call the measure m the weak limit measure of the harmonic function.

Proof. Note that m_r is indeed a Radon measure since $u(r,.) \in C/(T)$. We have

$$|m_r(f)| \leq \|f\|_\infty \int_0^1 |u(r\theta)|\, d\theta \leq C\|f\|_\infty$$

Letting r vary, the numbers $|m_r(f)|$ remain bounded, hence there is a sequence $r_k \to 1$ such that

$$m_{r_k} f$$

converges to a number c. We need to show that $m_r(f)$ converges to c not only along this sequence but genuinely for $r \to 1$.
Let v be the harmonic extension of f, since $v(r,.)$ converges uniformly to f as $r \to 1$, there is an r_ϵ such that for $r_\epsilon < r < 1$ we have

$$\|v(r,.) - f\|_\infty \leq \epsilon.$$

Let $r_\epsilon < r < 1$ and pick k such that $r < r_k < 1$ such that $|m_{r_k}(f) - c| < \epsilon$. Then we have

$$m_r(f) = \int_0^1 u(r,\theta)f(\theta)\, d\theta$$

$$= \int_0^1 \int_0^1 P(r/r_k, \theta - \phi) u(r_k, \phi) f(\theta)\, d\phi d\theta$$
$$= \int_0^1 \int_0^1 u(r_k, \phi) P(r/r_k, \phi - \theta) f(\theta)\, d\theta d\phi$$
$$= \int_0^1 u(r_k, \phi) v(r/r_k, \phi) d\phi.$$

And therefore

$$|c - m_r(f)| \le \epsilon + |m_{r_k}(f) - m_r(f)|$$
$$\le \epsilon + |\int_0^1 u(r_k, \theta)(f(\theta) - v(r/r_k, \theta))\, d\theta|$$
$$\le \epsilon + C\epsilon.$$

This proves convergence.
Define a function $m : C(\mathbb{T}) \to \mathbb{C}$ by

$$m(f) = \lim_{r\to 1} m_r(f)$$

The function m is linear since all m_r are linear, and we have the bound

$$|m(f)| \le \limsup_{r\to 1} |m_r(f)| \le C\|f\|_\infty$$

Hence m is a Radon measure with the desired bound on the total mass. □

Theorem 7. *The maps in theorems 5 and 6 are inverses of each other, i.e. the weak limit measure of the harmonic extension of a measure is the measure itself, and the harmonic extension of the weak limit measure is the original harmonic function.*

Proof. First assume we are given a Radon measure on $\mathbb{T}$.
Now assume we are given a harmonic function u with the bounds of theorem 6. Let m be the weak limit measure. The harmonic extension of m satisfies

$$m(P(r, \theta - .) = \lim_{s\to 1} m_s(P(r, \theta - .) = \lim_{s\to 1} u(rs, \theta) = u(r, \theta).$$

Now assume we are given a Radon measure m on $\mathbb{T}$ and let u be the harmonic extension. Let We have

$$u(r, \theta) = \sum_{n\in\mathbb{Z}} \widehat{m}(n) r^{|n|} e^{2\pi i n\theta}$$

=

$$\sum_{n\in\mathbb{Z}} m(e^{-2\pi i n.}) r^{|n|} e^{2\pi i n\theta} = m(\sum_{n\in\mathbb{Z}} r^{|n|} e^{2\pi i n(\theta - .)}).$$

Here have used uniform absolute convergence of the sum in the last expression to interchange the sum with evaluation of m. Note the argument of m in the last expression is the Poisson kernel. Hence we have

$$m_r(f) = \int_0^1 m(P(r, \theta - .)) f(\theta)\, d\theta$$

Using for example Riemann sums as in the proof of Theorem 5 we may interchange the integral with evaluation of m and obtain, using also symmetry of the Poisson kernel

$$m_r(f) = m(\int_0^1 P(r, . - \theta) f(\theta)\, d\theta)$$

Now the argument of m on the right hand side converges uniformly to f as r tends to 1, hence

$$\lim_{r\to 1} m_r(f) = m(f)$$

This completes the proof of the theorem.

□

The following theorem gives a simple criterion, when the previous theorem can be applied.

Theorem 8. *Let u be a positive harmonic function in the unit disc. Then u is the harmonic extension of some Radon measure with $\|m\|_{M^1} \le u(0)$.*

Proof. We check the criterion of Theorem 6. Since u is positive, we have by the mean value principle

$$\int_0^1 |u(r,\theta)| d\theta = \int_0^1 u(r,\theta) d\theta = u(0)$$

The theorem follows. □

Some Radon measures are given be integration against a continuous function. If g is a continuous funcion of the circle, then

$$m(f) := \int f(\theta) g(\theta)\, d\theta$$

defines a Radon measure. However, not all Radon measures are of this form. One example is

$$m(f) = f(0)$$

The Fourier series of this measure is constant, $\widehat{m}(n) = 1$, hence its harmonic extension is the Posson kernel. The Poisson kernel does not have a continuous extension to the boundary of the disc, hence m is not given as integration against a continuous function.

3. Lecture: L^2 functions and spectral measures

Let u be harmonic in the unit disc and $f_r(\theta) = u(r, \theta)$.
If u is the harmonic extension of a function in the Wiener algebra, we have uniformly in $0 \leq r < 1$

$$\sum_{n \in \mathbb{Z}} |\widehat{f_r}(n)| \leq C.$$

If u is the harmonic extension of a Radon measure, we have uniformly in $0 < r < 1$

$$\int_0^1 |f_r(\theta)| \, d\theta \leq C.$$

In neither situation we have very good information about the quantity decisive in the other situation. In the first situation we can deduce

$$\sup_\theta |f_r(\theta)| \leq C.$$

and in the second situation we can deduce

$$\sup_n |\widehat{f_r}(n)| \leq C.$$

but neither of these latter conditions are sufficient to guarantee we are in the respective situation.
Sometimes it is useful to have necessary and sufficient conditions for both f_r and $\widehat{f_r}$ to guarantee that one is in the same space.
One example is provided in the next theorem.

Theorem 9. *Let u be a harmonic function in the unit disc and f_r as above. Then we have*

$$\sum_{n \in \mathbb{Z}} |\widehat{f_r}(n)|^2 = \int_0^1 |f_r(\theta)|^2 \, d\theta.$$

If and only if this quantity is bounded uniformly in r, u is the harmonic extension of a Radon measure m on the boundary with

$$\sum_{n \in \mathbb{Z}} |\widehat{m}(n)|^2 < \infty.$$

Proof. Let $r < s$ and recall that

$$\widehat{f_r}(n) = \widehat{f_s}(n)(r/s)^{|n|}.$$

The function f_s is continuous on $\mathbb{T}$ and therfore has bounded Fourier coefficients, and therefore f_r is in the Wiener algebra. Writing f_r as Fourier series, we obtain

$$\int_0^1 |f_r(\theta)|^2 \, d\theta = \int_0^1 |\sum_{n \in \mathbb{Z}} \widehat{f_r}(n) e^{2\pi i n \theta}|^2 \, d\theta$$

$$\int_0^1 \sum_{n, n' \in \mathbb{Z}} \widehat{f_r}(n) \overline{\widehat{f_r}(n')} e^{2\pi i (n - n')\theta} \, d\theta = \sum_{n \in \mathbb{Z}} |\widehat{f_r}(n)|^2.$$

Here we have interchanged sum and integral thanks to the uniform absolute summability of the integrand.

Now assume that this quantity is bounded by C uniformly in r. We have

$$\int |f_r(\theta)|\, d\theta \leq \int_0^1 (1 + |f_r(\theta)|^2)\, d\theta \leq 1 + C.$$

Hence u is the harmonic extension of a Radon measure m by Theorem 6. The Fourier coefficients of m satisfy

$$r^{|n|}\widehat{m}(n) = \widehat{f_r}(n).$$

Since

$$\sum_n |\widehat{m}(n)|^2 r^{2|n|} \leq C$$

uniformly in r, we have by monotone convergence as $r \to 1$

$$\sum_n |\widehat{m}(n)|^2 \leq C.$$

Now assume conversely that u is the harmonic extension of a measure m with the last displayed inequality. then

$$\sum_n |\widehat{f_r}(n)|^2 = \sum_n r^{2|n|}|\widehat{m}(n)|^2 \leq \sum_n |\widehat{m}(n)|^2 \leq C.$$

This completes the proof of the theorem.

□

The theory of Lebesgue integration will allow to equate the quantity

$$\sum_n |\widehat{m}(n)|^2$$

in the previous theorem with a square integral over $\mathbb{T}$. We will discuss this later.
The space H of measures with the norm

$$\|m\|_2 = \sqrt{\sum_{n\in\mathbb{Z}} |\widehat{m}(n)|^2}$$

is a Hilbert space, i.e., a complete normed space with norm coming from a Hermitian inner product

$$\langle m, m'\rangle = \sum_{n\in\mathbb{Z}} m(n)\overline{m'(n)}$$

in the sense

$$\|m\|_2^2 = \langle m, m\rangle.$$

Here the attribute "Hermitian" means sesquilinearity

$$\langle f + \lambda g, h\rangle = \langle f, h\rangle + \lambda\langle g, h\rangle$$

$$\langle f, g + \lambda h\rangle = \langle f, g\rangle + \overline{\lambda}\langle f, h\rangle$$

for elements f, g, h in H and a complex number λ, and the relation

$$\langle f, g\rangle = \overline{\langle g, h\rangle}.$$

The norm of a Hilbert space satisfies the parallelogram law

$$2(\|g\|_2^2 + \|h\|_2^2) = \|g + h\|^2 + \|g - h\|^2,$$

as one can see by writing the norms in terms of the inner product and use binomial formulas. Conversely a complete normed space with the parallelogram law is a Hilbert space with the Hermitian form

$$\langle g,h\rangle = \frac{1}{4}(\|g+h\|^2 - \|g-h\|^2 + i\|g+ih\|^2 - i\|g-ih\|^2) \tag{9}$$

The proof of this is a bit more involved an left as an exercise. Equation(9) implies the Cauchy-Schwarz inequality

$$\langle g,h\rangle \le \|g\|\|h\|$$

As this inequality is invariant under multiplying g and h by complex scalars, we may assume the norms of g and h are 1 and the hermitian product of both on the left hand side is real and positive. Then the purely imaginary terms on the right hand side of (9) cancel and we obtain

$$\langle g,h\rangle \le \frac{1}{4}\|g+h\|^2 \le \frac{1}{4}(\|g\|+\|h\|)^2 = 1 = \|g\|\|h\|.$$

One other important fact about Hilbert spaces is the following theorem.

Theorem 10. *Let H be a Hilbert space and $\lambda : H \to \mathbb{C}$ a linear map that is bounded, i.e. there is a constant C such that*

$$|\lambda(f)| \le C\|f\|$$

for all $f \in H$. Then there is a $g \in H$ such that

$$\lambda(f) = \langle f, g\rangle$$

for all $f \in H$

Proof. If λ is constant 0, then the conclusion holds with $g = 0$. Assume λ is not constant zero, then there is a nonempty set

$$A = \{f \in H : \lambda(f) = 1\}.$$

By the C-bound on λ, we have

$$\inf_{f\in A} \|f\| \ge C^{-1}.$$

Multiplying λ and the desired g by a constant if necessary, we may assume

$$\inf_{f\in A} \|f\| = 1.$$

Let f_n be a minimzing sequence in A, i.e.,

$$\lim \|f_n\|_n = 1.$$

We compute for large n, n'

$$\|f_n - f_{n'}\|^2 = 2\|f_n\|^2 + 2\|f_{n'}\|^2 - \|f_n + f_{n'}\|^2$$

Note that $\lambda(\frac{1}{2}(f_n + f_{n'})) = 1$. Hence

$$\|f_n + f_{n'}\| \ge 2$$

Hence we have for small $\epsilon > 0$ and sufficiently large n, n':

$$\|f_n - f_{n'}\|^2 \le 2(1+\epsilon) + 2(1+\epsilon) - 4 = 4\epsilon$$

This shows that f_n is Cauchy in H and by completeness has a limit g. Then we have

$$\lambda(g) = 1 = \langle g, g\rangle$$

To show

$$\lambda(f) = \langle f, g\rangle$$

it suffices to show this for

$$f - \lambda(f)g$$

and hence, modifying f if necessary, it suffices to show this under the assumption

$$\lambda(f) = 0.$$

Consider

$$\lambda(g + af)$$

for a complex parameter a. Since $g + af \in A$ we have

$$1 \le \|g + af\|^2 = \|g\|^2 + 2\mathrm{Re}\langle af, g\rangle + \|af\|^2$$

We let θ in $a = re^{2\pi i\theta}$ so that $\mathrm{Re}\langle af, g\rangle = \langle af, g\rangle$. Then

$$1 \le 1 + 2r\langle f, g\rangle + r^2\|f\|^2.$$

The right hand side must have a minimum at $r = 0$ and thus

$$\langle f, g\rangle = 0.$$

This completes the proof of the theorem. □

Given a bounded linear operator $T : H \to H$, a consequence of the above is that there is an adjoint operator $T^* : H \to H$ such that for all f, g

$$\langle Tf, g\rangle = \langle f, T^*g\rangle.$$

(Exercise)
We use a Hilbert space to show a natural occurrence of harmonic extensions of Radon measures.
A unitary operator U on a Hilbert space is a linear map $U : H \to H$ such that

$$\|Ux\| = \|U^*x\| = \|x\|$$

for all $x \in H$. Equivalently $U^* = U^{-1}$. The adjoint of a unitary operator is unitary as well.
For $|z| < 1$, a unitary operator U and a vector $x \in H$ the series

$$\sum_{n\ge 0} \overline{z}^n U^n x$$

converges in H (here we have set $U^0x = x$ and $U^{n+1}x = U(U^nx)$), because

$$\|\overline{z}^n U^n x\| = |z|^n\|x\|$$

which tends to zero exponentially fast in n.
Likewise, the series

$$\sum_{n\geq 0} z^n U^{*n} x$$

converges, and hence the series

$$(1 + \sum_{n\geq 1} z^n U^{*n} + \sum_{n\geq 1} \overline{z}^n U^n)x$$

converges. In analogy with prior formulae for the Poisson kernel we write this as

$$P(zU^*)x$$

We claim that for every $x \in H$ the function

$$\langle P(zU^*)x, x\rangle \tag{10}$$

is a nonnegative real harmonic function in the unit disc. That it is harmonic follows from its representation as Fourier series

$$\langle x, x\rangle + \sum_{n\geq 1} z^n \langle U^{*n} x, x\rangle + \sum_{n\geq 1} \overline{z}^n \langle U^n x, x\rangle.$$

To see positivity for $x \neq 0$, note that the operator

$$Vx = \sum_{n\geq 0} \overline{z}^n U^n x$$

is an inverse to $1 - \overline{z}U$

$$(1 - \overline{z}U)Vx = V(1 - \overline{z}U)x = \sum_{n\geq 0} \overline{z}^n U^n x - \sum_{n\geq 1} \overline{z}^n U^n x = \overline{z}^0 U^0 x = x$$

By computations similar to that of the Poisson kernel, we obtain

$$\langle P(zU^*)x, x\rangle = \langle (1 - |z|^2)V^* V x, x\rangle,$$

for example expand the product V^*V on the right hand side and collect terms of each order. Set

$$y = Vx,$$

Then the previous display becomes

$$(1 - |z|^2)\langle y, y\rangle \geq 0.$$

Hence there is a Radon measure m_x such that

$$\langle P(zU^*)x, x\rangle = m_x(P(ze^{-2\pi i \cdot}) \tag{11}$$

Here we have written with our old convention about radial variables

$$P(ze^{-2\pi i\phi}) = P(r, \theta - \phi).$$

The measure m_x is called the spectral measure of U at x.
It is natural to try to define for an arbitrary continuous function f on the circle

$$\langle f(U)x, x\rangle := m_x(f) \tag{12}$$

More precisely, one defines with polarization a form

$$\langle f(U)x, y\rangle :=$$

$$= m_{x+y}(f) - m_{x-y}(f) + i m_{x+iy}(f) - i m_{x-iy}(f)$$

that is sesquilinear by the above exercise because the form (12) satisfies the analogue of the parallelogram law. This sesquilinear form, using Theorem 10, defines a bounded linear operator $f(U)$.
For two continuous functions f and g on $\mathbb{T}$ we have for every x

$$m_x(f+g) = m_x(f) + m_x(g)$$

and hence for every x, y

$$\langle (f+g)(U)x, y\rangle = \langle f(U)x, y\rangle + \langle g(U)x, y\rangle$$

and therfore

$$(f+g)(U) = f(U) + g(U).$$

We claim that also

$$(fg)(U) = f(U) \circ g(U), \tag{13}$$

where $\circ$ denotes composition of operators. To see this we first compute $f(U)$ for $f(z) = z^n$. We have that $m_x(f) = \widehat{m}(-n)$ by definition of the Fourier coefficients. Expanding the Poisson kernel in (11) gives

$$\langle x, x\rangle + \sum_{n\geq 1} z^n \langle U^{*n} x, x\rangle + \sum_{n\geq 1} \overline{z}^n \langle U^n x, x\rangle$$
$$= \widehat{m}_x(0) + \sum_{n\geq 1} z^n \widehat{m}_x(n) + \sum_{n\geq 1} \overline{z}^n \widehat{m}_x(-n)$$

Comparing coefficients we have

$$m_x(f) = \langle U^n x, x\rangle$$

Hence,

$$\langle f(U)x, x\rangle = \langle U^n x, x\rangle$$

and by polarization

$$\langle f(U)x, y\rangle = \langle U^n x, y\rangle$$

and thus $f(U) = U^n$. Similarly, if $f(z) = \overline{z}^n$, we obtain $f(U) = (U^*)^n$. If both f and g are monomials in Z or $\overline{z}$, then we obtain (13) by explicit calculation from the above representation. If both f and g are Laurent polynomials, that is linear combinations of monomials in z and monomials in $\overline{z}$, then (13) follows by linearity of both sides. Finally, the Stone-Weierstraß theorem tells us that every continuous function can be approximated in uniform (L^∞) norm by Laurent polynomials, so (13) follows by such approximation for arbitrary continuous functions f and g.
Similarly as the product pormula one proves

$$\overline{f}(U) = (f(U))^*.$$

If there is an orthonormal basis of eigenvectors of U (for example if H is finite dimensional), then one can concude from the sum and product formula as well as approximation by Stone and Weierstrass that the basis vectors are also eigenvectors of $f(U)$ and if one eigenvector of U

has the eigenvalue λ, then necessarily $|\lambda| = 1$ because U is unitary, and $f(U)$ has the eigenvalue $f(\lambda)$.
In general, that is in infinte dimensional Hilbert spaces, a unitary operator may not have an orthonormal basis of eigenfunctions. Then one does not have such an explicit description of $f(U)$ and has to resort to the represnetation by the spectral measures.

3.1. **Exercises.** Consider a real complete normed space, that is a real vector space V with norm $\|.\| : V \to \mathbb{R}$, that is for all $x, y \in V$ and $\lambda \in \mathbb{R}$ we have

$$\|x + y\| \le \|x\| + \|y\|$$
$$\|\lambda x\| = |\lambda| \|x\|,$$

and the metrix $d(x, y) = \|x - y\|$ is complete. Show that if the parallelogram law holds, that is for all $x, y \in V$ we have

$$2(x\|^2 + \|y\|^2) = \|x + y\|^2 + \|x - y\|^2,$$

then

$$\langle x, y\rangle := \frac{1}{2}(\|x + y\|^2 - \|x - y\|^2)$$

defines a symmetric scalar product, in particular

$$\langle x + \lambda y, z\rangle = \langle x, z\rangle + \lambda\langle y, z\rangle$$

for all $x, y, z \in V$ and $\lambda \in \mathbb{R}$
Hint: First show

$$\langle x + y, z\rangle + \langle x - y, z\rangle = 2\langle x, z\rangle$$

then show

$$\langle x, z\rangle + \langle y, z\rangle = \langle x + y, z\rangle$$

and finally

$$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$$

first for integers λ and then for general λ.

4. Lecture: The dyadic model of harmonic functions

A Radon measure on the circle corresponds to a harmonic function in the disc with certain estimates.
Each point in the disc comes with a particular Poisson kernel, which is some weighted average of the measure, with a weight which favors points on the circle with approximately the same angular variable as the point of the disc, and approximity measured roughly in terms of the distance of the point in the disc to the boundary.
We now introduce a discrete version of this harmonic function. We identify the circle with the interval $[0,1)$ as fundamental domain of $\mathbb{R}/\mathbb{Z}$ via the angular variable, functions on the circle become 1-periodic functions on the real line. In the discrete world, we take averages over socalled dyadic intervals in $[0,1)$.
A dyadic interval is an interval of the form

$$[2^k n, 2^k(n+1))$$

with integers k and n. The interval is inside the unit interval $[0,1)$ provided $k \leq 0$ and $0 \leq k < 2^{-k}$. We denote by $\mathcal{D}$ the set of dyadic intervals contained in $[0,1)$. We denote by $|I|$ the length of the dyadic interval I, that is the number 2^k in the above notation. We denote by $\mathcal{D}_k$ the dyadic intervals in $\mathcal{D}$ of length 2^k. We denote by $c(I)$ the center of the interval, $2^k(n+1/2)$.

Lemma 11. *For all $x \in [0,1)$ and all integers $k \leq 0$ there is a unique $I \in \mathcal{D}_k$ with $x \in I$.*

Proof. Fix $0 \leq x < 1$ and k a non-positive integer so that

$$0 \leq 2^{-k}x < 2^{-k}$$

with integral bounds on both sides. There is a unique integer n satisfying

$$0 \leq n \leq 2^{-k}x < n+1 \leq 2^{-k}, \tag{14}$$

namely the integer part of $2^{-k}x$. However, (14) for an integer n is equivalent to $I = [2^k n, 2^k(n+1)) \in \mathcal{D}_k$ and $x \in I$, proving the lemma. □

We write $I_{x,k}$ for the interval in this lemma. If $I \in \mathcal{D}$ and

$$I = [2^k n, 2^k(n+1)),$$

we write

$$I_l = [2^{k-1}(2n), 2^{k-1}(2n+1)),$$
$$I_r = [2^{k-1}(2n+1), 2^{k-1}(2n+2)).$$

The right endpoint of I_l is the left endpoint of I_r, so they are disjoint and their union is the interval between the left endpoint of I_l and the right endpoint of I_r, which is I. In particular, I is the disjoint union of I_l and I_r.

Lemma 12. *If $I, J \in \mathcal{D}$ and $I \cap J \neq \emptyset$, then $I \subset J$ or $J \subset I$.*

Proof. By symmetry between I and J we may assume $|I| \leq |J|$. The number $\log_2(|J|/|I|)$ is a natural number n. We prove by induction on n that $I \cap J \neq \emptyset$ implies $I \subset J$. If $n = 0$, then both intervals have the same length, say 2^k, and we use Lemma 11 to conclude that they are the same if they have a point x in common. Now assume we have proven the statement for some n and let $\log_2(|J|/|I|) = n + 1$. Consider J_l and J_r, we have $\log_2(|J_l|/|I|) = n$ and $\log_2(|J_r|/|I|) = n$ and in particular $|I| \leq |J_l|$ and $|I| \leq |J_r|$. Let x be a point in the intersection of I and J. Since J is the union of J_l and J_r, I has a point in common with J_l or J_r. By induction hypothesis, I is contained in J_l or J_r. But J_l and J_r are both subsets of J, hence I is a subset of J. This completes the induction step. □

Definition 13. *A martingale on $\mathcal{D}$ is a function $F : \mathcal{D} \to \mathbb{C}$ satisfying*

$$|I_l|F(I_l) + |I_r|F(I_r) = |I|F(I)$$

A martingale is a discrete version of a harmonic function. The following is a discrete version of the mean value property.

Lemma 14. *Let $I \in \mathcal{D}$ and let $2^k \leq |I|$. Let F be a martingale. Then*

$$\sum_{J \in \mathcal{D}_k : J \subset I} |J|F(J) = |I|F(I). \tag{15}$$

Proof. We have $2^k = 2^{-h}|I|$ with some nonnegative integer h. We do induction on h. If $h = 0$, the sum on the left hand side of (15) consist of a single summand $J = I$ and the identity is tautological. Assume we have proven the lemma for some h and consider $2^k = 2^{-h-1}|I|$. By Lemma (12), each J on the left hand side of (15) is either contained in I_l or in I_r. By induction hypothesis we have

$$\sum_{J \in \mathcal{D}_k : J \subset I} |J|F(J) = \sum_{J \in \mathcal{D}_k : J \subset I_l} |J|F(J) + \sum_{J \in \mathcal{D}_k : J \subset I_r} |J|F(J)$$

$$= |I_l|F(I_l) + |I_r|F(I_r) = |I|F(I),$$

the last identity by the martingale property. □

The next theorem will describe a discrete version of harmonic extension. Given a Radon measure m on $[0, 1)$, we would like to define a martingale, where $F(I)$ is an average of m over I. We have in mind

$$|I|^{-1} m(1_I)$$

with 1_I the characteristic function I, here we identify the function 1_I on $[0, 1)$ with a 1-periodic function on $\mathbb{R}$. Only if $I = [0, 1)$ is this periodic extension a continuous function, otherwise the last display is not yet defined. We will therefore approximate the characteristic function by continuous functions.

Assume $I \in \mathcal{D}$, say

$$I = [2^k n, 2^k(n + 1))$$

with $k < 0$. We consider the continuous 1-periodic function $\chi_{I,h}$ mapping $\mathbb{R} \to [0,1]$ and equal to 1 on the interval $[2^k n - 2^{k-h-2}, 2^k(n+1) - 2^{k-h-1})$ and its periodic copies and supported on the interval $[2^k n - 2^{k-h-1}, 2^k(n+1) - 2^{k-h-2})$ and its periodic copies. We further assume

$$\sum_{0 \le n < 2^{-k}} \chi_{[2^k n, 2^k(n+1)),h} = 1.$$

Such a function can for example be constructed as a piecewise linear spline.

Theorem 15. *Let m be a Radon measure on $\mathbb{T}$. Then the limit*

$$F(I) = |I|^{-1} \lim_{l \to \infty} m(\chi_{I,l}) \tag{16}$$

exists and defines a martingale with

$$\sum_{I \in \mathcal{D}_k} |I||F(I)| \le C < \infty. \tag{17}$$

uniformly in k. Moreover, for any $I \in \mathcal{D}$ and $\epsilon > 0$ there is $J \in \mathcal{D}$ with $J \subset I$ and $\overline{J} \notin I$ such that for all $k < 0$

$$\sum_{K \in \mathcal{D}_k : K \subset J} |K||F(K)| \le \epsilon. \tag{18}$$

The bound (17) is a discrete version of the condition

$$\int |u(r,\theta)| \, d\theta < C < \infty$$

satisfied by harmonic extensions of a measure. The property (18) is a technical condition casued by our choice to modify in (16) the charactristic functions on the left side of the boundary points. This choice is in sync with the choice that dyadic intervals are closed on the left and open on the right.

Proof. We first show that the limit in (16) exists. We write for $l \ge 1$

$$m(\chi_{I,l}) = m(\chi_{I,0}) + \sum_{h=1}^{l} m(\chi_{I,l} - \chi_{I,l-1}).$$

To see that this is Cauchy in l, it suffices to show

$$\sum_{1 \le h \le H} |m(\chi_{I,h}) - \chi_{I,h-1})| \le C < \infty$$

with constant C independent of H. Choose $|\lambda_h| = 1$ such that

$$m(\lambda_h(\chi_{I,h} - \chi_{I,h-1})) = |m(\chi_{I,h}) - \chi_{I,h-1})|$$

We obtain for the left hand side of the previous display

$$m(\sum_{1 \le h \le H} \lambda_h(\chi_{I,h} - \chi_{I,h-1})) \le \|m\|_{M^1} \| \sum_{1 \le h \le H} \lambda_h(\chi_{I,h} - \chi_{I,h-1}) \|_\infty$$

However, the summand

$$\lambda_h(\chi_{I,h} - \chi_{I,h-1}) \tag{19}$$

is non-zero on

$$(2^k n - 2^{k+h}, 2^k n - 2^{k+h-2}) \cup (2^k(n+1) - 2^{k+h}, 2^k(n+1) - 2^{k+h-2})$$

and the periodic copies of these sets. Note the first interval is to the left of $2^k n$ and the second interval is to the right of $2^k n$. For two different values h, h' these supports are disjoint if h and h' are more than one apart. Hence on every point x there are at most two of the functions (19) that are not zero. Since both functions are bounded in absolute value by 1, we have that

$$\| \sum_{1 \le h \le H} \lambda_h (\chi_{I,h}) - \chi_{I,h-1}) \|_\infty \le 2$$

Hence the limit in (16) exists.
To prove the martingale property, note that

$$\chi_{I_l,h} + \chi_{I_r,h} = \chi_{I,h+1}$$

Applying m and taking a limit as $h \to \infty$ proves the martingale property.
To see (17), we estimate

$$\sum_{I \in \mathcal{D}: |I| = 2^k} |I| |F(I)| \le 2 \sum_{I \in \mathcal{D}: |I| = 2^k} |m(\chi_{I,h})|$$

for suitably large h. We find suitable numbers λ_I such that the last display equals

$$= 2 \sum_{I \in \mathcal{D}: |I| = 2^k} m(\lambda_I \chi_{I,h}) = 2m(\sum_{I \in \mathcal{D}: |I| = 2^k} \lambda_I \chi_{I,h})$$

$$\le 2 \|m\|_{M^1} \| \sum_{I \in \mathcal{D}: |I| = 2^k} \lambda_I \chi_{I,h_I} \|_\infty .$$

At every point x at most two of the functions in the sum are nonzero, and each of the functions is bounded by 1, so we estimate the last display by

$$\le 4 \|m\|_{M^1}$$

This proves (17).
To see (18), we assume to get a contradiction that there is $I \in \mathcal{D}_k$ and $\epsilon > 0$ such that for each $J \subset I$, $\overline{J} \notin I$ there is k with

$$\sum_{K \in \mathcal{D}_k, K \subset J} |K| |F(K)| > \epsilon$$

Pick such I and ϵ and $J_1 = I$. Pick k_1 such that

$$\sum_{K \in \mathcal{D}_{k_1}, K \subset J_1} |K| |F(K)| > \epsilon$$

Pick h_1 large enough so that

$$\sum_{K \in \mathcal{D}_k, K \subset J_1} |m(\chi_{K,h_1})| > \epsilon.$$

Pick J_2 so that J_2 has a parent to the right of the support of all χ_{K,h_1}. Pick k_2 and h_2 so that

$$\sum_{K\in\mathcal{D}_{k_2},K\subset J_2} |m(\chi_{K,h_2})| > \epsilon$$

Now iterate this procedure, and note that all functions $\chi_{\dots}$ in the sum

$$\sum_{j=1}^{J} \sum_{K\in\mathcal{D}_{k_j},K\subset J_j} |m(\chi_{K,h_j})| > J\epsilon$$

are disjointly supported. Pick suitable λ_j so that

$$m(\sum_{j=1}^{J} \sum_{K\in\mathcal{D}_{k_j},K\subset J_j} \lambda_j\chi_{K,h_j})| > J\epsilon$$

Since the argument of m is bounded by 2 in ∞-norm, we have

$$\|m\|_{M^1} \geq J\epsilon$$

this is a contradiction for large enough J.

□

The next theorem is the dyadic version of Theorem 6.

Theorem 16. *Consider a martingale F such that there is a constant $C < \infty$ such that for all $k \leq 0$ we have* (17)*. Define*

$$m_k(f) = \sum_{I\in\mathcal{D}_k} |I|F(I)f(c(I))$$

Then the limit

$$m(f) := \lim_{k\to-\infty} m_k(f)$$

exists and defines a Radon measure.

Proof. We have

$$|m_k(f)| \leq \sum_{I\in\mathcal{D}_k} |I||F(I)|\|f\|_\infty \leq C\|F\|_\infty$$

by (17). We claim that for every f the limit $\lim_{k\to-\infty} m_k(f)$ exists. Let $\epsilon > 0$. As f is uniformly continuous, we may choose k small enough that for all intervals I of length at most 2^k and all $x, y \in I$ we have

$$|f(x) - f(y)| \leq \epsilon$$

Then for $k' < k$, we have using that every $I' \in \mathcal{D}_k$ is contained in a unique $I \in /Dd_k$ by Lemmas 11 and 12

$$|m_k(f) - m_{k'}(f)| \leq$$

$$|\sum_{I\in\mathcal{D}_k} (|I|F(I)f(c(I)) - \sum_{I'\in\mathcal{D}_{k'},I'\subset I} |I'|F(I')f(c(I')|)$$

The difference between $f(c(I'))$ and $f(c(I))$ for every pair $I' \subset I$ is at most ϵ. Using the discrete mean value property 14 we estimate this by

$$\leq \sum_{I\in\mathcal{D}_k} \sum_{I'\in\mathcal{D}_{k'}:I'\subset I} \epsilon|I'||F(I')| = \sum_{I'\in\mathcal{D}_{k'}} \epsilon|I'||F(I')| \leq \epsilon C.$$

This shows that the limit $\lim_{k\to-\infty} m_k(f)$ exists. The function m is linear and bounded using linearity and the uniform bound on the m_k.
$\square$

5. Lecture: Martingales and spectral resolution

Theorem 15 produced for each Radon measure on $\mathbb{T}$ a martingale

$$F(I) = |I|^{-1} \lim_{h\to\infty} m(\chi_{I,h}) \tag{20}$$

which satisfies for each k

$$\sum_{I\in\mathcal{D}_k} |I||F(I)| \le C \tag{21}$$

for some C independent of k and the technical condition that for all $I \in \mathcal{D}$ and al $\epsilon > 0$ there is a $J \in \mathcal{D}$ with $J \subset I$ and $\overline{J} \notin I$ such that for all k

$$\sum_{K\in\mathcal{D}_k : K\subset J} |K||F(K)| \le \epsilon \tag{22}$$

Conversely, Theorem 16 produced for every martingale satisfying a uniform bound (21) for all k a Radon measure

$$m(f) = \lim_{k\to-\infty} \sum_{I\in\mathcal{D}_k} f(c(I))|I|F(I) \tag{23}$$

The next theorem show that the maps of the two theorems are inverse of each other.

Theorem 17. *If m is a Radon measure on $\mathbb{T}$ and F the martingale extension* (20)*, then m and F satisfy* (23)*. If F is a martingale satisfying* (21) *and* (22) *and m is its weak limit* (23)*, then m and F satisfy* (20)*.*

Proof. Let m be a Radon measure on $\mathbb{T}$ and F the martingale extension (20). Fix f and let $\epsilon > 0$. It suffices to show that there is a constant C such that for sufficiently small k we have

$$\left| m(f) - \sum_{I\in\mathcal{D}_k} f(c(I))|I|F(I) \right| \le C\epsilon$$

Let k be sufficiently small so that $|f(x) - f(y)| \le \epsilon$ for $|x - y| \le 2^{k+2}$. Then for sufficiently large h,

$$\left| m(f) - \sum_{I\in\mathcal{D}_k} f(c(I))|I|F(I) \right|$$

$$\le \epsilon + |m(f) - \sum_{I\in\mathcal{D}_k} f(c(I)))m(\chi_{I,h})| = \epsilon + |m(f - \sum_{I\in\mathcal{D}_k} f(c(I))\chi_{I,h})|$$

But we have

$$\|f - \sum_{I\in\mathcal{D}_k} f(c(I))\chi_{I,h}\|_\infty \le 2\epsilon$$

because for fixed x

$$|f(x) - \sum_{I\in\mathcal{D}_k} f(c(I))\chi_{I,h}(x)|$$

$$\le \sum_{I\in\mathcal{D}_k} \chi_{I,h}(x)|f(x) - f(c(I))| \le 3\epsilon$$

The last inequality between the two lines following from the fact that $\chi_{I,h}(x)$ is boudned by 1 and non-zero only if I is equal to or neighboring to $I_{x,k}$. We obtain the desired

$$\left| m(f) - \sum_{I \in \mathcal{D}_k} f(c(I))|I|F(I) \right| \le 3\epsilon \|m\|_{M^1}$$

Now let F be a martingale satisfying (21) and (22). Let m be the weak limit in (23).
We compute for large h and small k

$$||I|F(I) - m(\chi_{I,h})| \le \epsilon + ||I|F(I) - \sum_{K \in \mathcal{D}_k} |K|F(K)\chi_{I,h}(c(K))|$$

Replacing $\chi_{I,h}(c(K))$ by the indicator $1_{K \subset I}$, we have by the martingale property

$$|I|F(I) - \sum_{K \in \mathcal{D}_k} |K|F(K)1_{K \subset I} = 0.$$

For most K, we have that $\chi_{I,h}(c(K))$ is $1_{K \subset I}$. Possible exceptions are those K that are contained in $J \cup J'$ with

$$J = [c(I) - |I|2^{-1} - |I|2^{-h-1}, c(I) - |I|2^{-1})$$
$$J' = [c(I) + |I|2^{-1} - |I|2^{-h-1}, c(I) + |I|2^{-1}).$$

here we assumed that k si small enough so that $2^k < 2^{-h-1}$.
We thus obtain

$$||I|F(I) - m(\chi_{I,h})| \le \epsilon + 2 \sum_{K \in \mathcal{D}_k : K \subset J} |K||F(K)| + 2 \sum_{K \in \mathcal{D}_k : K \subset J'} |K||F(K)|$$

By property (18), the last display is bounded by 3ϵ for sufficiently large h. In the limit, we obtain the desired (20) □

Consider a unitary operator U and recall the spectral measures m_x defined by

$$m_x(P(ze^{-2\pi i \cdot}) = \langle P(zU^*)x, x \rangle$$

and the operator $f(U)$ for continuous function f on $\mathbb{T}$ defined by

$$\langle f(U)x, y \rangle = \frac{1}{4}(m_{x+y}(f) - m_{x-y}(f) + im_{x+iy}(f) - im_{x-iy}(f)). \tag{24}$$

Putting $f = \chi_{I,h}$ and taking the linit as $h \to \infty$, which exists by Theorem (15) for the right-hand-side, we obtain the limit

$$\lim_{h \to \infty} \langle \chi_{I,h}(U)x, y \rangle$$

this is a sesquilinear form since each $\langle \chi_{I,h}(U)x, y \rangle$ is sesquilinear, so we obtain an operator $1_I(U) : H \to H$ defined by

$$\langle 1_I(U)x, y \rangle := \lim_{h \to \infty} \langle \chi_{I,h}(U)x, y \rangle.$$

Since the operators $\chi_{I,h}(U)$ are self adjoint and bounded, so are the operators $1_I(U)$.

Let I and J be two disjoint dyadic intervals. Since $1_I 1_J = 0$, we expect $1_I(U) \circ 1_J(U) = 0$. However, we have proven the product formula only for contiunuosu functions, hence we argue carefully. We have

$$\langle 1_I(U) \circ 1_J(U)x, y\rangle = \langle 1_J(U)x, 1_I(U)y\rangle$$

Assume without loss of generality that J is to the left of I. Fix $\epsilon > 0$ and let h be large enough so that

$$|\langle \chi_{J,h}(U)x, 1_I(U)y\rangle - \langle 1_J(U)x, 1_I(U)y\rangle| \le \epsilon.$$

Pick l large enough so that $\chi_{J,h}$ is supported to the left to $\chi_{I,l}$ and

$$|\langle \chi_{J,h}(U)x, 1_I(U)y\rangle - \langle \chi_{J,h}(U)x, \chi_{I,l}(U)y\rangle| \le \epsilon$$

Since

$$\langle \chi_{J,h}(U)x, \chi_{I,l}(U)y\rangle = \langle (\chi_{I,l}\chi_{J,h})(U)x, y\rangle = 0,$$

we have

$$\langle 1_J(U)x, 1_I(U)y\rangle| \le 2\epsilon.$$

Since ϵ and x and y were arbitrary, $1_I(U) \circ 1_J(U) = 0$. Taking adjoints, we conclude $1_J(U) \circ 1_I(U) = 0$.

Now let $I \subset J$ with I strictly smaller than J. As $1_I 1_J = 1_I$, we expect $1_I(U) \circ 1_J(U) = 1_I(U)$ and provide an argument similar to the above. By standard properties of dyadic intervals we have $I \subset J_l$ or $I \subset J_r$. In the first case let h be large enough such that

$$|\langle \chi_{J,h}(U)x, 1_I(U)y\rangle - \langle 1_J(U)x, 1_I(U)y\rangle| \le \epsilon.$$

and let l be large enough such that $\chi_{I,l}$ is supported on the set where $\chi_{J,h}$ is equal to one and

$$|\langle \chi_{J,h}(U)x, 1_I(U)y\rangle - \langle \chi_{J,h}(U)x, \chi_{I,l}(U)y\rangle| \le \epsilon$$

Since

$$\langle \chi_{J,h}(U)x, \langle \chi_{I,l}(U)y\rangle = \langle \chi_{J,h}\chi_{I,l}(U)x, y\rangle = \langle \chi_{I,l}(U)x, y\rangle$$

and ϵ and x, y were arbitrary we obtain

$$1_J(U) \circ 1_I(U) = 1_I(U).$$

If $I \subset J_r$, we may similarly first choose the approximation of 1_I and then the approximation of 1_J to make sure that $\chi_{J,h}\chi_{I,l} = \chi_{I,l}$ and we obtain again $1_J(U) \circ 1_I(U) = 1_I(U)$.

Using the martingale property we obtain

$$1_I(U) = 1_{I_l}(U) + 1_{I_r}(U)$$

composing with $1_{I_l}(U)$ gives

$$1_{I_l}(U) \circ 1_I(U) = 1_{I_l}(U) \circ 1_{I_l}(U) + 1_{I_l}(U) \circ 1_{I_r}(U)$$

and by the previous observations

$$1_{I_l}(U) = 1_{I_l}(U) \circ 1_{I_l}(U)$$

Similarly $1_{I_r}(U) = 1_{I_r}(U) \circ 1_{I_r}(U)$. Since every interval other than $[0, 1)$ is left or right half of some other interval, we obtain that every $1_I(U)$

with $[0,1) \neq I$ is idempotent and self adjoint, and hence a projection operator.
In particular, we note for such projection operators

$$\|1_I(U)x\|^2 = \langle 1_I(U)x, 1_I(U)x\rangle = \langle 1_I(U)x, x\rangle \leq \|1_I(U)x\|\|x\|$$

We conclude

$$\|1_I(U)x\| \leq \|x\|,$$

this is clear if $\|1_I(U)x\|$ is zero, otherwise we divide the previous display by $\|1_I(U)x\|$.
For $I = [0,1)$ we observe that 1_I constant eqaul to 1 and hence

$$\langle 1_I(U)x, x\rangle = m_x(1) = m_x(P(0)) = \langle P(0)x, x\rangle = \langle x, x\rangle$$

Hence $1_{[0,1)}$ is the identity operator and in particular also a projection.
Using Theorem 16 to associate with m_x a martingale F_x and Theorem 7, we conclude for each k

$$\langle f(U)x, x\rangle = m_x(f) = \lim_{k\to-\infty} \sum_{I\in\mathcal{D}_k} |I|F_x(I)f(c(I))$$

$$= \lim_{k\to-\infty} \sum_{I\in\mathcal{D}_k} \langle 1_I(U)x, x\rangle f(c(I)) = \lim_{k\to-\infty} \langle \sum_{I\in\mathcal{D}_k} f(c(I))1_I(U)x, x\rangle$$

Indeed, this convergence is in operator norm.

$$\|T\|_{op} = \sup_{\|x\|=1} \|Tx\|$$

Theorem 18 (Spectral resolution)**.** *For a unitary operator, we have*

$$f(U) = \lim_{k\to-\infty} \sum_{I\in\mathcal{D}_k} f(c(I))1_I(U)$$

In the sense

$$\lim_{k\to-\infty} \|f(U) - \sum_{I\in\mathcal{D}_k} f(c(I))1_I(U)\|_{op}.$$

Proof. Let x be of norm *one*, then we have for $k' < k$ by the martingale property

$$\|(\sum_{I\in\mathcal{D}_k} f(c(I))1_I(U) - \sum_{I'\in\mathcal{D}_{k'}} f(c(I'))1_{I'}(U))x\|^2$$

$$= \|(\sum_{I\in\mathcal{D}_k} \sum_{I'\in\mathcal{D}_{k'}:I'\subset I} (f(c(I)) - f(c(I'))1_I(U))x\|^2$$

By orthogonality of the projections, expaning the inner product, and by unioform contonity of f,

$$= \sum_{I\in\mathcal{D}_k} \sum_{I'\in\mathcal{D}_{k'}:I'\subset I} \|(f(c(I)) - f(c(I'))1_I(U)x\|^2$$

$$\leq \epsilon \sum_{I\in\mathcal{D}_k} \sum_{I'\in\mathcal{D}_{k'}:I'\subset I} \|1_I(U))x\|^2$$

$$= \epsilon\| \sum_{I\in\mathcal{D}_k} \sum_{I'\in\mathcal{D}_{k'}:I'\subset I} 1_I(U))x\|^2$$

$$= \epsilon\|x\|^2 = \epsilon$$

Hence the sequence $\sum_{I\in\mathcal{D}_k} f(c(I))1_I(U)$ is Cauchy in the operator norm. That it converges to $f(U)$ follows form the weak calculations in ahead of stating the theorem.

□

In particular, since for $f(\theta) = e^{2\pi i\theta}$ we have $f(U) = U$

$$U = \lim_{k\to-\infty} e^{2\pi i c(I)} 1_I(U)$$

We make the following observation in finite dimensional Hilbert spaces.

Theorem 19. *Let U be a unitary operator on a finite dimensional Hilbert space H. Then there exists a basis of H consisting of eigenvectors of H.*

Proof. For each interval $I \in \mathcal{D}$, pick $x_I = 0$ if $1_I(U) = 0$ and otherwise pick x_I in the range of $1_I(U)$, that is $x = 1_I(U)x$.
The collection

$$A_k = \{x_I : I \in \mathcal{D}_k, x_I \neq 0\}$$

consists of pairwise orthogonal vectors since for I, I' disjoint we have

$$\langle x_I, x_{I'}\rangle = \langle 1_I(U)x_I, 1_{I'}(U)x_{I'}\rangle = 0$$

Therefore, these vectors are linearly independent,from

$$\sum_{I\in A_k} \lambda_I x_I \neq 0$$

we conclude by pairing with x_J that $\lambda_J = 0$. Hence there are at most as many vectors in A_k as the dimension of H.
The cardinality of A_k is weakly increasing as $k \to -\infty$ as for each $1_I(U) \neq 0$ we have by the martingale property that $1_{I_l}(U) \neq 0$ or $1_{I_r}(U) \neq 0$. Since the cardinality of A_k is bounded, it stabilizes at a certain k_0, so the cardinality is constant for each $k \geq k_0$ Thus we have for each $|I| \leq 2^{k_0}$ with $1_I(U) \neq 0$ that precisely one of $1_{I_l}(U) \neq 0$ or $1_{I_r}(U) \neq 0$ holds. More precisely, we have by the martingale property $1_I(U) = 1_{I_l}(U)$ or $1_I(U) = 1_{I_r}(U)$.
Enumerate the intervals in A_{k_0} as $J_1, \dots, J_n$. For each n and each $k < k_0$ there is a unique interval $J_{k,n} \in A_k$ such that

$$1_{J_n}(U) = 1_{J_{k,n}}(U)$$

. For fixed n, the sequence of closed intervals $\overline{J_{k,n}}$ has unique intersection point θ_n.
Let $f(\theta) = e^{2\pi i\theta}$ We have

$$U = \lim_{k\to-\infty} \sum_{I\in\mathcal{D}_k} f(c(I))1_I(U)$$

$$= \lim_{k\to-\infty} \sum_n f(c(J_{k,n}))1_{I_n}(U)$$

$$= \sum_n f(\theta_n)1_{I_n}(U) + \lim_{k\to-\infty} \sum_n (f(c(J_{k,n})) - f(\theta_n))1_{I_n}(U)$$

The limit in the last display tends to zero as for sufficiently large k by continuity of f, similarly to thre proof of the last theorem.
Hence

$$f(U)x = \sum_n f(\theta_n) 1_{I_n}(U)x$$

Thus on the range of each $1_{I_n}(U)$, U acts as a multiplication by the constant (eigenvalue) $f(\theta_n)$. The projections 1_{I_n} are pairwise orthogonal, and the sum is the identity, therefore the sum of ranges spans all of H. Picking an orthonormal basis for each range, the union of these bases is an orthonormal basis of eigenvectors of U.

□

6. Lecture: Decomposition theorems

Consider the space $\mathcal{S}^\Delta(\mathbb{T})$ of finite linear combinations of characteristic functions of dyadic intervals in $\mathcal{D}$. Decomposing all dyadic intervals in such a linear combination into small intervals of equal length, we may assume that such as function is of the form

$$f = \sum_{I \in \mathcal{D}_k} f_I 1_I$$

for some k and some coefficients f_I. Write $\mathcal{S}_k^\Delta$ for the set of functions of this form. The largest k such that $f \in \mathcal{S}_k^\Delta$ is called the scale of f. By the martingale property, for $f \in \mathcal{S}_k^\Delta$ and any martingale F, the sequence

$$\lim_{l \to -\infty} \sum_{I \in \mathcal{D}_l} |I| F(I) f(c(I))$$

stabilizes once $l \leq k$. In particular, the sequence converges. We write $m(f)$ for the limit if F is the martingale extension of a Radon measure m. The functional $m(f)$ is linear in f.

Lemma 20. *For every Radon measure m on $\mathbb{T}$ with martingale extension F we have*

$$\|m\|_{M^1} = \sup_{f \in \mathcal{S}^\Delta(\mathbb{T}) : \|f\|_\infty \leq 1} m(f).$$

Proof. Let $\epsilon > 0$ be given. Pick $g \in C(\mathbb{T})$ with $\|g\|_\infty = 1$ such that

$$\|m\|_{M^1} \leq \epsilon + |m(g)|.$$

Pick further k small enough so that

$$|m(g)| \leq \epsilon + \Big| \sum_{I \in \mathcal{D}_k} |I| F(I) g(c(I)) \Big|.$$

Then define $f \in \mathcal{S}_k^\Delta$ such that $f_I := g(c(I))$ on intervals $I \in \mathcal{D}_k$ and note

$$\|f\|_\infty \leq \|g\|_\infty = 1,$$

$$\|m\|_{M^1} \leq 2\epsilon + |m(f)|.$$

Since ϵ was arbitrary,

$$\|m\|_{M^1} \leq \sup_{f \in \mathcal{S}^\Delta(\mathbb{T}) : \|f\|_\infty \leq 1} |m(f)|.$$

To see the reverse inequality, let again $\epsilon > 0$ be given. and let $f \in \mathcal{S}^\Delta(\mathbb{T})$ be given with $\|f\|_\infty \leq 1$. Observe that for sufficiently large k and h

$$|m(f)| = \Big| \sum_{I \in \mathcal{D}_k} |I| F(I) f(c(I)) \Big| \leq \epsilon + \Big| \sum_{I \in \mathcal{D}_k} m(\chi_{I,h}) f(c(I)) \Big|$$

$$\leq \epsilon + \|m\|_{M^1} \Big\| \sum_{I \in \mathcal{D}_k} \chi_{I,h} f(c(I)) \Big\|_\infty$$

$$\leq \epsilon + \|m\|_{M^1} \|f\|_\infty \Big\| \sum_{I \in \mathcal{D}_k} \chi_{I,h} \Big\|_\infty = \epsilon + \|m\|_{M^1}.$$

Since f and ϵ were arbitrary, this shows the reverse inequality. $\square$

We have the following consequence of the previous lemma.

Lemma 21. *Two Radon measures m_1, m_2 are equal, if for every $f \in \mathcal{S}^\Delta(\mathbb{T})$ we have $m_1(f) = m_2(f)$.*

Proof. Consider the Radon measure $m = m_1 - m_2$. Then $m(f) = 0$ for all $f \in \mathcal{S}^\Delta(\mathbb{T})$. By the previous lemma, $\|m\|_{M^1} = 0$ and hence $m = 0$ and $m_1 = m_2$. □

In what follows, we pass to the real setting. A Radon measure m is called real if for all real valued $f \in \mathcal{S}^\Delta(\mathbb{T})$ we have that $m(f)$ is real. For a Radon measure m, define for real functions f the real part $m_1(f) := \mathrm{Re}(m(f))$ and the imaginary part $m_2(f) := \mathrm{Im}(m(f))$ and extend m_1 and m_2 to complex valued functions by linearity. Then m_1 and m_2 are real and we have $m = m_1(f) + im_2(f)$. The measures m_1 and m_2 are the unique real Radon measures such that $m = m_1(f) + im_2(f)$. Our next goal is to decompose a real Radon measure into positive and negative part.
A Radon measure m is positive if for all nonnegative real $f \in \mathcal{S}^\Delta(\mathbb{T})$ we have $m(f) \geq 0$. Note that for a positive Radon measure we have for all $\|f\|_\infty \leq 1$

$$|m(f)| = |\lim_{k\to-\infty} \sum_{I \in \mathcal{D}_k} |I| F(I) f(c(I))| \leq \lim_{k\to-\infty} \sum_{I \in \mathcal{D}_k} |I| F(I)| = m(1)$$

and thus $m(1) = \|m\|_{M^1}$. Recall that a characteristic function is a function that only takes values 0 and 1.

Definition 22. *Two Radon measures are called mutually singular, $m_1 \perp m_2$, if for every $\epsilon > 0$ there exist two characteristic functions χ_1 and χ_2 in $\mathcal{S}^\Delta$ with $\chi_1 + \chi_2 = 1$ such that for every $f \in \mathcal{S}^\Delta$ we have*

$$|m_2(f\chi_1)| \leq \epsilon \|f\|_\infty,$$
$$|m_1(f\chi_2)| \leq \epsilon \|f\|_\infty.$$

Lemma 23. *Let m_1, m_2 be mutually singular and $m = m_1 + m_2$. Let $\epsilon > 0$ be given. Pick χ_1 and χ_2 as in the definition of mutual singularity. Then we have for $j = 1, 2$ and every $f \in \mathcal{S}^\Delta$,*

$$|m(f\chi_j) - m_j(f\chi_j)| \leq \epsilon \|f\|_\infty$$
$$|m_j(f\chi_j) - m_j(f)| \leq \epsilon \|f\|_\infty$$

Moreover, we have

$$\|m\|_{M^1} = \|m_1 - m_2\|_{M^1}$$

Proof. Assume without loss of generality $j = 1$. The first inequality follows from

$$m(f\chi_1) - m_1(f\chi_1) = m_2(f\chi_1),$$

while the second inequality follows from

$$m_1(f\chi_1) - m_1(f) = m_1(f\chi_2).$$

To see the last inequality, pick $\epsilon > 0$. Let f with $\|f\|_\infty = 1$ such that

$$\|m\|_{M^1} \le |m(f)| + \epsilon = |m(f\chi_1) + m(f\chi_2)| + \epsilon$$

$$\le |m_1(f\chi_1) + |m_2(f\chi_2)| + 3\epsilon \le |(m_1 - m_2)(f\chi_1 - f\chi_2)| + 5\epsilon.$$

As χ_1 and χ_2 have disjoint support, we have

$$\|f\chi_1 + f\chi_2\|_\infty = \|f\chi_1 - f\chi_2\|_\infty = 1$$

As ϵ was arbitrary, we conclude

$$\|m\|_{M^1} \le \|m_1 - m_2\|_{M^1}$$

The reverse inequality follows by the same argument applied to $-m_2$ in place of m_2. This completes the proof of the lemma. □

We have the following decomposition theorem.

Theorem 24. *Any real Radon measure m can be decomposed as*

$$m = m_+ - m_-$$

with two unique mutually singular positive measures m_+ and m_-.

Proof. Let F be the martingale extension of m. Define for positive $f \in \mathcal{S}^\Delta$:

$$m_+(f) = \lim_{k \to -\infty} \sum_{I \in \mathcal{D}_k : F(I) \ge 0} |I| F(I) f(c(I))$$

$$m_-(f) = -\lim_{k \to -\infty} \sum_{I \in \mathcal{D}_k : F(I) < 0} |I| F(I) f(c(I))$$

To see existence of the limits of these positive sequences, we establish that the sequences are eventually monotone increasing and bounded. Monotonicity follows by the martingale property

$$F(I) 1_{F(I) \ge 0} = F(I_l) 1_{F(I) \ge 0} + F(I_r) 1_{F(I) \ge 0}$$

$$\le F(I_l) 1_{F(I_l) \ge 0} + F(I_r) 1_{F(I_r) \ge 0}.$$

Hence, once k is smaller than the scale of f, both sequences in these limits are monotone increasing. Both sequences are bounded by

$$\sum_{I \in \mathcal{D}_k} |I| |F(I)| \|f\|_\infty \le \|m\|_{M^1} \|f\|_\infty$$

hence the limits exists and are bounded by the right-hand-side of the last display. By linearity, the limits defining m_+ and m_- continue to exist for $f \in \mathcal{S}^\Delta$ that is not necessarily positive, and thus we obtain positive Radon measures m_+ and m_- with $m = m_+ - m_-$. To see that the measures m_+ and m_- are mutually singular, let $\epsilon > 0$ be given. Pick h small enough so that

$$\|m\|_{M^1} - \epsilon \le \sum_{I \in \mathcal{D}_h : F(I) \ge 0} |I| |F(I)|$$

Let χ_+ be the characteristic function of the union of those $I \in \mathcal{D}_h$ with $F(I) \geq 0$ and let $\chi_+ + \chi_- = 1$. Then

$$\|m\|_{M^1} - \epsilon \leq \sum_{I \in \mathcal{D}_h} |I| F(I) \chi_+(c(I) - \sum_{I \in \mathcal{D}_h} |I| F(I) \chi_-(c(I).$$

Note that for $k < h$ the right hand side is by the martingale identity equal to

$$\sum_{J \subset I : F(J) \geq 0, F(I) \geq 0} |J| F(J) + \sum_{J \subset I : F(J) \leq 0, F(I) \geq 0} |J| F(J)$$

$$- \sum_{J \subset I : F(J) \leq 0, F(I) \leq 0} |J| F(J) - \sum_{J \subset I : F(J) \geq 0, F(I) \leq 0} |J| F(J),$$

where in each sum it is understood that $J \in \mathcal{D}_k, I \in \mathcal{D}_h$. By collecting positive and negative terms, we see that this is equal to

$$\sum_J |J||F(J)| - 2 \sum_{J \subset I : F(J) \leq 0, F(I) \geq 0} |J||F(J)| - 2 \sum_{J \subset I : F(J) \geq 0, F(I) \leq 0} |J||F(J)|$$

As the first term is bounded by $\|m\|_{M^1}$, we have

$$\sum_{J \subset I : F(J) \leq 0, F(I) \geq 0} |J||F(J)| + \sum_{J \subset I : F(J) \geq 0, F(I) \leq 0} |J||F(J)| \leq \epsilon.$$

Hence for $f \in \mathcal{S}^\Delta$ with $\|f\|_\infty \leq 1$ and k smaller than h and smaller than the scale of f

$$m_+(f \chi_-) \leq \sum_{J \subset I : F(J) \geq 0, F(I) < I} |J||F(J)| \leq 2\epsilon$$

and similarly

$$m_-(f \chi_+) \leq 2\epsilon.$$

This proves that m_+ and m_1 are mutually singular.
To prove uniqueness, let $\tilde{m}_+, \tilde{m}_-$ be a different mutually singular pair of positive measures with $m = \tilde{m}_+ - \tilde{m}_-$. It suffices to show that for every $\epsilon > 0$ we have $\|\tilde{m}_+ - m_+\|_{M^1} \leq C\epsilon$.
Let $\epsilon > 0$ be given and define χ_+ and χ_- as above so that

$$\|m\|_{M^1} - \epsilon \leq m(\chi_+ - \chi_-)$$

Observe by positivity the following two inequalities:

$$\tilde{m}_+(\chi_+ - \chi_-) \leq \tilde{m}_+(\chi_+),$$

$$-\tilde{m}_-(\chi_+ - \chi_-) \leq \tilde{m}_-(\chi_-).$$

Adding the last two and inserting into the previous gives

$$\|m\|_{M^1} - \epsilon \leq \tilde{m}_+(\chi_+) - \tilde{m}_-(\chi_-) \leq \|\tilde{m}_+\|_{M^1} + \|\tilde{m}_-\|_{M^1} = \|m\|_{M^1}$$

All inequalities must be sharp within ϵ, so that

$$\tilde{m}_+(\chi_+) \geq \|\tilde{m}_+\| - \epsilon$$

Testing $\chi_+ + \lambda f \chi_-$ for $|\lambda| = 1$ and any $\|f\| \leq 1$ and using the first inequality gives

$$|\tilde{m}_+(f \chi_-)| \leq \epsilon$$

and similar
$$|(m-\tilde{m}_+)f\chi_+| = |\tilde{m}_-(f\chi_+)| \le \epsilon$$
The measure m_+ satisfies the same estimates, we see that
$$|(m_+-\tilde{m}_+)f| \le |(m_+-\tilde{m}_+)(f\chi_+)| + |(m_+-\tilde{m}_+)(f\chi_-)| \le 4\epsilon.$$
This proves uniqueness.

□

Define a partial order on real measures by $m \le \tilde{m}$ if for all nonnegative functions $m_1(f) \le m_2(f)$. In particular, for positive and negative part of m,
$$m_- \le m \le m_+,$$
and for any positive measure m we have $0 \le m$.

Lemma 25. *If $m \le \tilde{m}$, then $m_+ \le \tilde{m}_+$ and $m_- \ge \tilde{m}_-$.*

Proof. By additivity, it suffices to prove the first inequality. Recall
$$m_k(f) = \sum_{I\in\mathcal{D}_k : F(I)\ge 0} |I|F(I)f(c(I))$$
from the proof of the previous lemma. It suffices to prove $(m_k)_+ \le (\tilde{m}_m)_+$ for every k. But $m \le \tilde{m}$ implies $F(I) \le \tilde{F}(I)$, from which the desired estimates follows. □

Lemma 26. *Consider two positive measures m_1 and m_2. Then the limit*
$$\lim_{N\to\infty}(m_1 - Nm_2)_+$$
exists and is a positive measure.

The limit measure is called the singular part of m_1 relative to m_2.

Proof. The sequence is monotone decreasing seqeunce of positive measures. Hence for each nonnegative $f \in \mathcal{S}^\Delta$ the sequence
$$(m_1 - Nm_2)_+(f)$$
is a decreasing sequence of nonnegative numbers and thus converges. The limit defines a positive measure . □

A frequent example concerns m_2 the Lebesgue measure, when the limit of the theorem is called genuinely the singular part of m_1.
$$m_2(f) = \int f(\theta)\, d\theta$$
If m_1 is integration against a real continuous function, then for sufficiently large N the measure $m_1 - Nm-2$ is integration against a negative continuous function, which is the negative of a positive measure and thus has vanishing positive part. Hence the singular part of m_1 is zero. However the singular part of m_1 the Dirac delta is m_1 itself:
$$(m_1 - Nm_2)_+(f) = \lim_{k\to\infty} 2^k F([0,2^k))f(2^{k-1})$$

$$= \lim_{k\to\infty}(1 - N2^k)f(2^{k-1}) = f(0).$$

The measure

$$m_1 - \lim_{N\to\infty}(m_1 - Nm_2)_+$$

is called the absolutely continuous part of m_1 relative to m_2.

7. Lecture: Martingale average convergence

We consider the singular and absolutely continuous part of a positive Radon measure on $[0,1)$ with respect to Lebesgue measure

$$1(f) = \int_0^1 1(x)f(x)\,dx.$$

We call a positive measure singular or absolutely continuous respectively if it is its own singular or absolutely continuous part. In this section, we study possible limits $\lim_{k\to-\infty} F(I_{x,k})$ and will establish a connection between such limits and the absolutely continuous part of a measure.

For a Radon measure m and $x \in [0,1)$ define the dyadic Hardy-Littlewood maximal function

$$Mm(x) = \sup_{k\le 0} |F(I_{x,k})|$$

If m is positive, we can omit the absolute value signs. The value $Mm(x)$ may be infinite for some x.

We will in this section denote by F, F_1 etc. the martingale extensions of m, m_1 etc.

Theorem 27. *Let m be a positive Radon measure on $[0,1)$. Let $N > 0$ and let $\mathcal{I}$ be the collection of maximal (with respect to set inclusion) dyadic intervals such that $F(I) > N$. Then*

$$\{x : Mm(x) > N\} \subset \bigcup \mathcal{I},$$

$$\sum_{I\in\mathcal{I}} |I| \le \frac{\|m\|_{M^1}}{N}.$$

Proof. If $Mm(x) > N$, then there is an interval $I_{x,k}$ with $F(I_{x,k}) > N$. There is a maximal such interval, wich will be in $\mathcal{I}$. This proves the first claim. To see the second claim, it suffices to show for every finite subcollection $\mathcal{I}'$ of $\mathcal{I}$ that

$$\sum_{I\in\mathcal{I}'} |I| \le \frac{\|m\|_{M^1}}{N}.$$

Let f be the sum of characteristic functions of intervals in $\mathcal{I}'$, then $f \in \mathcal{S}^\Delta$. Moreover, f is a characteristic function, because the intervals in $\mathcal{I}$, none of which can be contained in another by maximality, are all disjoint. We obtain

$$\|m\|_{M^1} \ge m(f) = \sum_{I\in\mathcal{I}'} m(1_I) = \sum_{I\in\mathcal{I}'} |I|F(I) \ge \sum_{I\in\mathcal{I}'} |I|N.$$

This proves the theorem. □

Definition 28. *We say a property holds for almost every $x \in [0,1)$, if for every $\epsilon > 0$ there exists a collection of dyadic intervals $\mathcal{I}$ with $\sum_{\mathcal{I}} |I| \le \epsilon$ such that the property holds for all x not in the set $\bigcup \mathcal{I}$.*

Theorem 29. *Let m be a positive Radon measure on $[0,1)$. Then for almost every $x \in [0,1)$ we have $Mm(x) < \infty$.*

Proof. For $N > \|m\|_{M^1}/\epsilon$ consider the collection $\mathcal{I}$ of maximal (with respect to set inclusion) dyadic intervals such that $|F(I)| > N$. By the previous theorem,

$$\sum_{I \in \mathcal{I}} |I| \le \|m\|_{M^1}/N \le \epsilon$$

Now assume $Mm(x) = \infty$, then there exists an x with $x \in I$ such that $F(I) > N$ and hence x is contained in the union of $\mathcal{I}$. This proves the theorem. □

Theorem 30. *A positive Radon measure m on $[0,1)$ is absolutely continuous if and only if for all $\delta > 0$ there exists $\epsilon > 0$ such that whenever $\mathcal{I}$ is a collection of intervals with*

$$\sum_{I \in \mathcal{I}} |I| \le \epsilon$$

then

$$\sum_{I \in \mathcal{I}} |I||F(I)| \le \delta$$

Proof. First assume m is absolutely continuous. Let δ be given. By approximating the nonnegative sum

$$\sum_{I \in \mathcal{I}} |I||F(I)|$$

by finite subsums, it suffices to prove the statement of the theorem for finite sets $\mathcal{I}$. Let 2^{k_0} be the smallest length of an interval in the finite set $\mathcal{I}$. As m is absolutely continuous, its singular part vanishes. Hence

$$\lim_{N \to \infty} \lim_{k \to -\infty} \sum_{I \in \mathcal{I}_k : |F(I)| - N > 0} |I||F(I) - N| = 0$$

Pick N large enough so that

$$\lim_{k \to -\infty} \sum_{I \in \mathcal{I}_k : |F(I)| > N} |I||F(I) - N| \le \delta/4$$

Pick $k < k_0$ small enough so that

$$\sum_{I \in \mathcal{I}_k : |F(I)| > N} |I||F(I) - N| \le \delta/2$$

Pick $\epsilon < \delta/(2N)$. Then

$$\sum_{I \in \mathcal{I}} |I||F(I)| \le \sum_{I \in \mathcal{D}_k, I \subset \bigcup \mathcal{I}} |I||F(I)|$$

$$\le \sum_{I \in \mathcal{D}_k, I \subset \bigcup \mathcal{I},} |I|N + \sum_{I \in \mathcal{D}_k, I \subset \bigcup \mathcal{I}, F(I) > N} |I||F(I) - N| \le N\epsilon + \delta/2 \le \delta$$

This proves the only if part theorem.
To see the if part, assume the δ-ϵ condition of the theorem is satisfied. Let $\delta > 0$ and pick $\epsilon > 0$ as in the condition of the theorem. Let N large enough so that for the collection $\mathcal{I}$ of maximal dyadic intervals with

$F(I) > N$ satisfies $\sum_{\mathcal{I}} |I| < \epsilon$. Then we have for $N' > N$ and k small enough

$$\|m\|_{sg} \le \|(m-N)_+\| \le \delta + \sum_{I \in \mathcal{I}_k : |F(I)| - N > 0} |I||F(I) - N| \le \delta + \sum_{\mathcal{I}} |I| F(I) \le 2\delta.$$

As δ was arbitrary, we conclude $m_{sg} = 0$. This completes the proof of the theorem. □

Theorem 31. *Let m be a positive Radon measure and F its martingale extension. Fix $0 \le a < b$ and $\epsilon > 0$. Then there exists $\mathcal{I} \subset \mathbb{D}$ and $k_0 < 0$ such that*

$$\sum_{I \in \mathcal{I}} |I| \le \epsilon,$$

and for all $x \notin \bigcup \mathcal{I}$ we have $F(I_{x,k}) < b$ for all $k < k_0$ or we have $F(I_{x,k}) \ge b$ for all $k < k_0$.

Proof. Let $\mathcal{I}_0$ be the set of maximal dyadic intervals I such that $F(I) \ge b$. Then by the argument of the previous theorem

$$\sum_{I \in \mathcal{I}_0} |I| \le \frac{\|m\|_{M^1}}{b} < \infty.$$

Now assume we have defined $\mathcal{I}_j$ for some $j \ge 0$. Let $\mathcal{J}_{j+1}$ be the set of maximal dyadic intervals J such that $J \subset I$ for some $I \in \mathcal{I}_m$ and $F(J) < a$.

Similarly let $\mathcal{I}_{j+1}$ be the set of maximal dyadic intervals I such that $I \subset J$ for some $J \in \mathcal{J}_{j+1}$ and $F(I) \ge b$. By disjointness of the intervals in $\mathcal{J}_{j+1}$

$$\sum_{J \in \mathcal{J}_{j+1}} |J| \le \sum_{I \in \mathcal{I}_j} |I|.$$

Further, we have by the bounds on the martingale values $F(I)$ and $F(J)$ and by the martingale identity

$$b \sum_{I \in \mathcal{I}_{j+1}} |I| \le \sum_{I \in \mathcal{I}_{j+1}} F(I)|I| \le \sum_{J \in \mathcal{J}_{j+1}} F(J)|J| < a \sum_{J \in \mathcal{J}_{j+1}} |I|.$$

Hence

$$\sum_{I \in \mathcal{I}_{j+1}} |I| < \frac{b}{a} \sum_{J \in \mathcal{J}_{j+1}} |J|.$$

Iterating this inequality gives

$$\sum_{I \in \mathcal{I}_j} |I| \le (\frac{a}{b})^j \sum_{I \in \mathcal{I}_0} |I|$$

For j large enough. We obtain

$$\sum_{I \in \mathcal{I}_j} |I| \le \frac{\epsilon}{2}.$$

The finitely many collections $\mathcal{I}_i$ and $\mathcal{J}_i$ for $i \le j$ may be infinite, albeit their sums of length of intervals is finite. Pick therefore respective finite subcollections $\tilde{I}_i$ and $\tilde{J}_i$ such that

$$\sum_{i=0}^{j} \sum_{I \in \mathcal{I}_j \setminus \tilde{\mathcal{I}}_j} |I| + \sum_{i=1}^{j} \sum_{J \in \mathcal{J}_j \setminus \tilde{\mathcal{J}}_j} |J| \le \frac{\epsilon}{2}.$$

Pick k_0 such that 2^k_0 is smaller than the length of any interval on the collections $\tilde{I}_i$ or all $\tilde{J}_i$ with $i \le j$.
Assume

$$x \notin \bigcup \mathcal{I}_j \cup \bigcup_{i=0}^{j} \mathcal{I}_j \setminus \tilde{\mathcal{I}}_j \cup \bigcup_{i=1}^{j} \mathcal{J}_j \setminus \tilde{\mathcal{J}}_j$$

where we call the collection of all intervals on the right hand side $\mathcal{I}$ and note $\sum_{\mathcal{I}} |I| < \epsilon$.
Then by the nesting of our collections of sets, one of the following three possibilities attains: First, we may have

$$x \notin \bigcup \mathcal{I}_0$$

In this case $F_{I_{x,k}} \le b$ for all $k \le 0$. Second, we may have for some $0 \le i < j$

$$x \in \bigcup \mathcal{I}_i \setminus \bigcup \mathcal{J}_{i+1}$$

and hence

$$x \in \bigcup \tilde{\mathcal{I}}_i \setminus \bigcup \mathcal{J}_{i+1}$$

In this case, there is a $k_0 \le k_1$ so that the interval I_{x,k_1} is in $\tilde{\mathcal{I}}_i$ and we have $F_{I_{x,k}} \le b$ for all $k \le k_1$.
In the third case, we have for some $1 \le i \le j$

$$x \in \bigcup \mathcal{J}_j \setminus \bigcup \mathcal{I}_i$$

and hence

$$x \in \bigcup \tilde{\mathcal{J}}_j \setminus \bigcup \mathcal{I}_i$$

In this case there is a $k_0 \le k_1$ such that the interval I_{x,k_1} is in $\mathcal{J}_i$ and we have $F_{I_{x,k}} \ge a$ for all $k \le k_1$. In all three cases we have proven the theorem.

□

Theorem 32. *Let m be Radon on $[0,1)$ and F its martingale extension. Let $\epsilon > 0$. Then there exist $\mathcal{I} \subset \mathcal{D}$ with $\sum_{I \in \mathcal{I}} |I| \le \epsilon$ and $k \le 0$ such that for all $k', k'' < k$ and all $x \notin \bigcup \mathcal{I}$ we have*

$$|F(I_{x,k'}) - F(I_{x,k''})| \le 2\epsilon.$$

Proof. Pick N large enough such that $\tilde{I}$, the set of maximal dyadic intervals I such that $F(I) > N$, satisfies

$$\sum_{\tilde{I}} |I| \le \frac{\|m\|_{M^1}}{N} \le \frac{\epsilon}{2}.$$

For every integre $0 \le n \le N/\epsilon$, we pick $\mathcal{I}_n$ and k_n so that

$$\sum_{I \in \mathcal{I}_n} |I| \le \epsilon^{-n-2}$$

and for all $x \notin \bigcup \mathcal{I}_n$ we have $F(I_{x,k}) < (n+1)\epsilon$ for all $k < k_n$ or we have $F(I_{x,k}) \ge n\epsilon$ for all $k < k_n$. Let $\mathcal{I} = \tilde{\mathcal{I}} \cup \bigcup_{0 \le n \le N/\epsilon} \mathcal{I}_n$. By adding a geometric series we have

$$\sum_{I \in \mathcal{I}} |I| \le \epsilon$$

Define further

$$k = \min_{0 \le n \le N/\epsilon} k_n.$$

Let $x \notin \bigcup \mathcal{I}$. As x is not in $\tilde{\mathcal{I}}$, we have for every $k \le 0$ that

$$0 \le F(I_{x,k}) \le N$$

Let n be the maximal integer such that

$$n\epsilon \le F(I_{x,k'})$$

for all $k' < k$. By the previous display, $n \le N/\epsilon$. As x is not in $\mathcal{I}_{n+1}$, we have

$$F(I_{x,k'}) \le (n+2)\epsilon$$

for all $k' < k$. Given two $k', k'' < k$, we conclude

$$|F(I_{x,k'}) - F(I_{x,k''})| \le 2\epsilon.$$

This proves the theorem. □

Theorem 33. *Let m be positive Radon measure with martingale extension F. Then the limit*

$$\lim_{k \to -\infty} F(I_{x,k})$$

exists almost everywhere.

Proof. Let $\epsilon > 0$ be given. We apply Theorem 32 to m consecutively with parameters $\epsilon 2^{-j}$ to obtain exceptional sets $\mathcal{I}_j$. Setting $I = \bigcup_j \mathcal{I}_j$, we obtain by summing a geometric series $sum_{\mathcal{I}}|I|\langle\epsilon$.Moreover, we have convergence of the limits $\lim_{k \to -\infty} F(I_{x,k})$ for every $x \notin \bigcup \mathcal{I}_j$. This proves the theorem. □

Theorem 34. *Let m_1 and m_2 be absolutely continuous and assume for all $\epsilon > 0$ there exists a collection $\mathcal{I}$ of dyadic intervals such that $\sum_{\mathcal{I}} |I| \le \epsilon$ and $x \notin \bigcup \mathcal{I}$*

$$\lim_{k \to \infty} |F_1(I_{x,k}) - F_2(I_{x,k}| = 0$$

Then $m_1 = m_2$.

Proof. By considering $m_2 - m_1$ if necessary, we may assume $m_1 = 0$. Let $\epsilon > 0$. Pick δ small and a collection $\mathcal{I}$ and a k such that $\sum_{I\in\mathcal{I}} |I| \le \delta$ and for all $k', k'' \le k$ and all $x \notin \bigcup \mathcal{I}$ we have

$$|F_2(I_{x,k'}) - F_2(I_{x,k''}| \le \epsilon.$$

As $F_2(I_{x,k''})$ converges to 0 as $k'' \to -\infty$, we have for all $x \notin \bigcup I$

$$F_2(I_{x,k'}) \le \epsilon$$

Now let for small k'

$$\|m_2\|_{M^1} \le \epsilon + \sum_{I\in\mathcal{D}_{k'}} |I||F_2(I)| \le \sum_{I\in\mathcal{D}_{k'} I\notin\mathcal{I}} |I||F_2(I)| + \sum_{I\in\mathcal{D}_{k'} I\subset\mathcal{I}} |I||F_2(I)|$$

The first summand on the right-hand-side is bounded by ϵ because w ecan bound $F(I)$ by ϵ. The second summand is bounded by ϵ, provided δ is small enough since m_2 is absolutely continuous. As ϵ was arbitrary, we conclude $\|m_2\|_{M^1} = 0$ and hence $m_2 = 0$.

□

Absolutely continuous measures are represented by their "always everywhere" boundary values. This leads to the theory of Lebesgue integrable (L^1) functions.

Theorem 35. *Let m be a singular positive Radon measure on $[0, 1)$. Then for almost every x in $[0, 1)$ we have*

$$\lim_{k\to\infty} F(I_{x,k}) = 0$$

Proof. Let $\epsilon > 0$ be given and pick N sufficiently large so that for the set $\mathcal{I}_1$ of maximal dyadic intervals such that $F(I) > N$ so that $\sum_{\mathcal{I}_1} |I| \le \epsilon$. As m is absolutely continuous, $m = (m - N)_+$ for all N and we have

$$\lim_{k\to-\infty} \sum_{I\in\mathcal{D}_k : F(I) > N} F(I)|I| = \|m\|_{M^1}$$

$$\lim_{k\to-\infty} \sum_{I\in\mathcal{D}_k : F(I) < N} F(I)|I| = 0$$

Pick k_0 small enough so that for all $k < k_0$

$$\sum_{I\in\mathcal{D}_k : F(I) < N} F(I)|I| \le \epsilon^2$$

Pick $\mathcal{I}_2$ be the maximal collection of dyadic intervals of length at most 2^{k_0} such that $F(I) > \epsilon$ We claim

$$\sum_{\mathcal{I}_2} |I| \le \epsilon$$

It suffices to prove the claim for a finite sub-collection $\mathcal{I}_2'$ of $\mathcal{I}_2$, Let 2^{k_1} be the smallest scale of this finest sub-collection. By the martingale identity we have

$$\epsilon \sum_{\mathcal{I}_2'} |I| \le \sum_{\mathcal{I}_2'} |I| F(I) = \sum_{I\in\mathcal{D}_{k_1} I\subset\bigcup\mathcal{I}_2'} |I| F(I) \le \epsilon^2$$

This proves the claim. It follows that for $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ we have $\sum_I |I| \le 2\epsilon$. For $x \notin \bigcup \mathcal{I}$, we have for all $k \le k_0$

$$F(I_x, k) \le \epsilon.$$

Repeat this argument with $\epsilon 2^{-j}$ fro all $j \ge 0$. The union of exceptional sets is smaller than 4ϵ , while outside the exceptional set we have

$$\lim_{k\to\infty} F(I_{x,k}) = 0$$

This proves the theorem.

□

7.1. **Exercise.** Let m, m' be Radon and F, G their martingale extension. Assume m absolutely continuous and let G be bounded martingale, i.e.

$$\|G\|_\infty = \sup_{I\in\mathcal{D}} |G(I)| < \infty.$$

Then

$$\lim_{k\to-\infty} \sum_{|I|=2^k} |I| F(I) G(I)$$

exists.