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Due on Friday 26 June.

Problem 1 (Walsh model)**.** Define

$$w_{2^i} = \sum_{I \in \mathcal{D}_i} (1_{I^l} - 1_{I^r}).$$

For each positive integer j, consider the binary expansion $j = \sum_{i=0}^{\infty} b_i 2^i$ where $b_i \in \{0, 1\}$, and let

$$w_j = \prod_{i=0}^{\infty} w_{2^i}^{b_i}.$$

(a) Prove that $\{w_j\}_j$ as defined above coincides with the $\{W_n = w([0,1) \times [n, n+1))\}_n$ defined in the lecture notes.

(b) Suppose that $I \times \omega$ is a tile with $\omega = 2^k[n, n+1)$ (height n). Show that

$$w(I \times \omega) = 1_I w_n(2^k \cdot).$$

(c) Prove that for each $n \in \mathbb{N}$, there is a unique $n' \in \mathbb{N}$ such that $W_{n'}$ has exactly n zero crossings.

Problem 2 (Fourier series)**.** (a) Verify the formula for the Dirichlet kernel

$$D_N := \sum_{j=-N}^{N} e^{2\pi i j t} = \frac{\sin(\pi(2N+1)t}{\sin(\pi t)}.$$

(b) Prove the Riemann–Lebesgue lemma: If $f \in L^1(\mathbb{T})$, then $\widehat{f}(k) \to 0$ when $|k| \to \infty$.

(c) Prove the Riemann localization principle: If f is zero in an open neighborhood of x, then

$$\lim_{N \to \infty} (D_N * f)(x) = 0.$$