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Due on Friday 19 June.

Problem 1 (Schwartz class)**.** For $f \in C^\infty(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{N}^n$ (natural numbers with zero included), define

$$|f|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$. We call f a Schwartz function if $|f|_{\alpha,\beta} < \infty$ for all multi-indices α and β.

(a) Show that the following operations preserve the Schwartz class: finite summation, pointwise product, multiplication by any polynomial, differentiation to any order.

(b) We say $f_n \to f$ as Schwartz functions if $|f_n - f|_{\alpha,\beta} \to 0$ for all α and β. Show that the Schwartz class is complete with respect to this convergence. Show that the convergence as Schwartz functions implies convergence in L^p norm for all $p \in [1, \infty]$.

(c) Show that the Schwartz class is dense in all $L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$.

(d) Show that the Fourier transform

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx$$

of a Schwartz function is again a Schwartz function.

(e) Show that $e^{-\pi|x|^2}$ is its own Fourier transform. Use this together with the heat extension from the first exercise sheet to conclude the Fourier inversion formula for Schwartz functions

$$f(x) = \int_{\mathbb{R}^n} (\mathcal{F}f)(\xi) e^{2\pi i \xi \cdot x} \, d\xi$$

Problem 2 (Riesz transform)**.** Let f be a Schwartz function on $\mathbb{R}^n$.

(a) Show that the limit

$$\lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{f(y)(x_i - y_i)}{|x-y|^{n+1}} \, dy$$

exists for $i \in \{1, \ldots, n\}$.

(b) Prove that for all $x \in \mathbb{R}^n$

$$f(x) = \frac{1}{|\partial B(0,1)|} \int \frac{\nabla f(y) \cdot (x-y)}{|x-y|^n} \, dy,$$

where the integral can be shown to exist as a principal value as above.

(c) Let $I_1 f(x) = \int f(y)|x-y|^{1-n} \, dy$. Show that

$$\Delta(I_1 I_1 f)(x) = c f(x)$$

for a constant c only depending on the dimension n.