Summer term 2020 | University of Bonn
Real and harmonic analysis | Prof. Dr. Christoph Thiele
Exercise sheet 7 | Dr. Olli Saari

Due on Friday 5 June. No exercise session on Monday June 1 (holiday).

Problem 1 (Hilbert transform)**.** Let m be a real valued Radon measure on the unit circle with $\|m\|_2 < \infty$ and $\widehat{m}(0) = 0$. We define its Hilbert transform through the identity

$$\widehat{Hm}(k) := -i \operatorname{sgn}(k)\widehat{m}(k)$$

where $\operatorname{sgn}(k) = 0$ if $k = 0$ and $\operatorname{sgn}(k) = k/|k|$ otherwise.

(a) Show that $f(re^{2\pi i\theta}) = m_r(\theta) + i(Hm)_r(\theta)$ is analytic in the interior of the unit disk and that $(Hm)_r$ is real valued. Here m_r and $(Hm)_r$ are the respective Poisson extensions. Conclude that

$$\int_0^1 f(re^{2\pi i\theta})^k \, d\theta = 0$$

for all integers $k \geq 2$.

(b) Let $k \geq 2$ be an even integer. Show that $\|Hm\|_{L^k} \leq C_k\|m\|_{L^k}$ for some C_k that does not depend on m when m is a smooth function. Hint: expand the real part of the above identity and use Hölder's inequality to separate L^k norms of m and Hm.

(c) Let $2 \leq p < \infty$. Show that $\|Hm\|_{L^p} \leq C_k\|m\|_{L^p}$ when m is a smooth function on the circle.

(d) Let $\tilde{m}(x) = m(-x)$. Show that for all smooth functions m and n we have $\int(Hm)n = \int(H\tilde{n})\tilde{m}$. Use duality between L^p and $L^{p'}$, $1/p+1/p' = 1$, and the fact that C^∞ is dense in $L^{p'}$ to show that $\|Hm\|_{L^p} \leq C_k\|m\|_{L^p}$, $1 < p < 2$.

Problem 2 (BMO and dyadic BMO)**.** Given a martingale extension $F : \mathcal{D} \to \mathbb{C}$ of a Radon measure m on the unit interval, recall that m is in dyadic BMO if

$$\sup_{\mathcal{E}} \left(\frac{1}{|I_{\mathcal{E}}|} \sum_{I\in\mathcal{E}} |I||\Delta F(I)|^2 \right)^{1/2} < \infty$$

when $\mathcal{E}$ is a tree with top interval $I_{\mathcal{E}}$. It is in BMO if $\sum_{k\in\mathbb{Z}} 1_{[0,1)}(k + \cdot + \tau)m(k + \cdot + \tau)$ is in dyadic BMO for all $\tau \in [0, 1)$.

(a) Let $I_0 \subset [0, 1)$ be dyadic. Show that if $\mathcal{E} = \{I \in \mathcal{D} : I \subseteq I_0\}$, then

$$\lim_{k\to-\infty} \frac{1}{|I_0|} \sum_{I\in\mathcal{D}_k\cap\mathcal{E}} |I||F(I) - F(I_0)|^2 = \frac{1}{|I_0|} \sum_{I\in\mathcal{E}} |I||\Delta F(I)|^2.$$

(b) Show that a 1-periodic $f \in L^1([0, 1))$ is in BMO if and only if

$$\sup_{I\subset\mathbb{R}} \frac{1}{|I|} \int_I \left| f - \frac{1}{|I|} \int_I f \right|^2 < \infty$$

where the supremum is taken over *all* intervals.

(c) Use the formula above to show that there exists a locally integrable function which is of dyadic BMO but not of genuine BMO type (hint: the candidate more or less already appeared in the lectures).