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Due on Friday 29 May.

Problem 1 (Hölder's inequality via tensor power trick)**.** Let $1 < p_1, \dots, p_k < \infty$ with $\sum_i 1/p_i = 1$ and $f_i \in \ell^{p_i}$ be sequences with values in $[0, \infty)$.

(a) Show that $a_1 \cdots a_k \leq \max_i a_i^{p_i} \leq \sum_i a_i^{p_i}$ for any numbers $a_1, \dots, a_k \geq 0$.

(b) Using part a show that

$$\sum_x \prod_{i=1}^k f_i(x) \leq k \prod_i \|f_i\|_{p_i}.$$

Hint: consider first the case $\|f_i\|_{p_i} = 1$ for all i.

(c) Using part b show that

$$\sum_x \prod_{i=1}^k f_i(x) \leq \prod_i \|f_i\|_{p_i}.$$

Hint: apply part b to the seqences $f_i^{\otimes m}(x_1, \dots, x_m) = f_i(x_1) \cdots f_i(x_m)$.

Problem 2 (Marcinkiewicz interpolation, classical version)**.** Fix $0 < p_0 < p_1 < \infty$, and let $T : L^{p_0}(\mathbb{R}) + L^{\infty}(\mathbb{R}) \to L^{p_0}(\mathbb{R}) + L^{\infty}(\mathbb{R})$ be a mapping that is

- sublinear: for all f and g in the domain $|T(f+g)| \leq |Tf| + |Tg|$
- homogeneous: for all constants c it holds $|T(cf)| = |c||Tf|$

and satisfies

$$\lambda^{p_0} |\{|Tf| > \lambda\}| \leq C_0 \int |f|^{p_0} \, dx$$
$$\lambda^{p_1} |\{|Tf| > \lambda\}| \leq C_1 \int |f|^{p_1} \, dx$$

for fixed constants C_0 and C_1, all $\lambda > 0$ and all functions $f \in L^{p_0} + L^{\infty}$. Prove that for each $p \in (p_0, p_1)$ there is a constant C_p only depending on C_0, C_1 and p such that

$$\int |Tf|^p \, dx \leq C_p \int |f|^p \, dx$$

holds for all measurable f such that the right hand side is finite.