University of Bonn
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Due on Friday 24 April 2020. Solve the exercises alone or in a group of two. Type the solutions in **tex**. Name the .tex file as

name_password.tex

where *password* is replaced by a password (10 symbols, say, letters and numbers) that will later give access to your corrected solution. Send the .tex file to the teaching assistant by email on the due day or earlier. The subject line of the email must be *real_and_harmonic_2020*.
Make sure your solution sheet contains your email address. The exercise session will be organized using Zoom on the Monday following the due day at 8:30, and the login data to the meeting will be provided to the email address(es) in the exercise sheets that have been handed in.

Problem 1 (Heat semigroup)**.** Let $\phi_{\sqrt{t}}(x) = (4\pi t)^{-1/2} \exp(-x^2/4t)$.

(a) Show that the function $\Phi(x,t) = \phi_{\sqrt{t}}(x)$ solves the heat equation $\partial_t F = \partial_x^2 F$ on $\mathbb{R} \times (0, \infty)$.

(b) Let f be a bounded Lipschitz function. Recall that the convolution of two functions is defined by $\phi * f(x) = \int_{\mathbb{R}} \phi(y) f(x-y) \mathrm{d}y$. Show that the function

$$F(x,t) := \begin{cases} \phi_{\sqrt{t}} * f(x), & t > 0, \\ f(x), & t = 0 \end{cases}$$

is continuous on $\mathbb{R} \times [0, \infty)$ and solves the heat equation on $\mathbb{R} \times (0, \infty)$.

Problem 2 (Continuous function with non-summable Fourier coefficients)**.** In this exercise, we study 1-periodic functions which we identify with functions defined on $[0, 1)$. Let $a \in (0, 1/2)$ and define

$$1_{(0,a)}(x) = \begin{cases} 1, & \text{if } 0 < x < a \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the Fourier coefficients of $1_{(0,a)}$. Show that they are not absolutely summable.

(b) Consider a continuous function $f_a : [0, 1] \to \mathbb{R}$ satisfying

$$f_a'(x) = 1_{(0,a)}(x) - 1_{(0,a)}\left(x - \frac{1}{2}\right), \quad x \notin \left\{0, a, \frac{1}{2}, \frac{1}{2} + a\right\}.$$

Prove upper and lower bounds for $\widehat{f_a}(n)$ (see the last item).

(c) Summing multiples of the function above, construct a continuous function whose Fourier coefficcients are not absolutely summable.