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Partial Differential Equations and Modelling

Sheet Nr.8 Due: 23.06.2017

Exercise 1

Prove that

$$\lambda^{\frac{d}{2}}\psi(\lambda^2 t, \lambda x)$$

satisfies the Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

on $\mathbb{R} \times \mathbb{R}^d$ if $\lambda > 0$ and ψ satisfies the Schrödinger equation and that

$$e^{-it|v|^2 + iv\cdot x}\psi(t, x - 2vt)$$

satisfies the Schrödinger equation if ψ does. Derive a formula for the solution with the initial data

$$\psi(0, x) = (2\pi\lambda^2)^{-\frac{d}{2}} e^{-\frac{1}{2\lambda^2}|x|^2 + i\xi\cdot x}.$$

Exercise 2

Derive an equation for $g(t, r)$ so that

$$u(t, x) = g(t, |x|)p_m(x/|x|)$$

satisfies the Harmonic oscillator equation if g satisfies that equation. Here p_m is a harmonic polynomial of degree m in $\mathbb{R}^d$.

Exercise 3

Describe all homogeneous harmonic polynomials of degree 0, 1 and 2 in $\mathbb{R}^d$. Find a basis for the homogeneous harmonic polynomials of degree 3 and 4 in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Exercise 4

The Schrödinger equation for a charged particle in a constant electric field in $d = 1$ is

$$i\partial_t u + u_{xx} - xu = 0$$

Prove that $u \to -u_{xx} + xu$ defines a selfadjoint operator for $D(-\partial_{xx} + x)$ containing $\mathcal{S}(\mathbb{R})$ by defining a unitary evolution using the Fourier transform. Describe the orbits for the Hamilton equations for the Hamilton function

$$H(p, x) = |p|^2 + x.$$