Prof. Dr. Herbert Koch
Dr. Xian Liao
Summer Term 2017

Partial Differential Equations and Modelling

Sheet Nr.5 Due: 26.05.2017

Exercise 1

Define the domain $D(-\Delta)$ of the operator $-\Delta$ to be

$$H^2(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \partial^2_{ij} f \in L^2(\mathbb{R}^d) \quad \text{for } i,j = 1,\dots,d\}.$$

Show that $-\Delta$ is closed, symmetric, selfadjoint and positive semidefinite.
Hint: This is easier if you use the Fourier transform.

Exercise 2

Let p be a real polynomial in d variables. Prove that

$$p(-iD)$$

defines a selfadjoint operator. Here

$$p(-iD) = \sum_{|\alpha|\le N} a^\alpha (-i)^{|\alpha|} \partial^\alpha,$$

where a^α are the constant coefficients of the polynomial $p(x) = \sum_{|\alpha|\le N} a^\alpha x^\alpha$, $x \in \mathbb{R}^d$.

Exercise 3

Let $U(t)$ be the rotation by the angle t in $\mathbb{R}^2$. Verify that $t \to U(t)$ is a unitary group and determine its generator as well as the domain of its generator. *Needs material from Wednesday*.

Exercise 4

Find a domain for $\frac{d}{dx} + x$ on $\mathbb{R}$ so that it is a closed operator.
Hint: The domain is

$$D(\frac{1}{i}\frac{d}{dx}) \cap D(M_x) = \{f \in L^2(\mathbb{R}) : xf, f' \in L^2(\mathbb{R})\}.$$

Expand the square of the norm.