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Partial Differential Equations and Modelling

Sheet Nr.3 Due: 12.05.2017

Exercise 1

Let $H = l^2(\mathbb{Z})$ be the set of square summable sequences. Let (e_j) be the standard basis and define $T(e_j) = e_{j+1}$. Determine the spectrum of T.

Exercise 2

Let $L^2(\mathbb{S}^1)$ be the set of the square integrable 2π periodic functions. Let H be the span of $\{e^{ikx}, k \geq 0\}$ and let Π be the orthogonal projection from $L^2(\mathbb{S}^1)$ to H. Given a bounded function h in H we define the Hankel operator T_h as

$$H \ni f \to T_h f = \Pi(f\bar{h}).$$

a) Determine the matrix corresponding to T_h in the basis (e^{ikx}).

b) Determine the spectrum and eigenfunctions of $T_{e^{ix}}$.

Exercise 3

Let

$$L\phi = -\Delta\phi + |x|^2\phi = f$$

for $\phi \in \mathcal{S}(\mathbb{R}^d)$. Prove that there exists $c(d)$ such that

$$\|\phi\|_{L^2}^2 + \||x|\phi\|_{L^2}^2 + \sum_{j=1}^d \|\partial_j\phi\|_{L^2}^2 + \||x|^2\phi\|_{L^2}^2 + \sum_{i,j=1}^d \|x_i\partial_j\phi\|_{L^2}^2 + \sum_{i,j=1}^d \|\partial_{ij}\phi\|_{L^2}^2 \le c(d)\|f\|_{L^2}^2.$$

Hint: Use $L = \sum_{j=1}^d(-\partial_{x_j} + x_j)(\partial_{x_j} + x_j) + d$ and integrate by parts first in $\langle L\phi, \phi\rangle$.

Exercise 4

Define a suitable Hilbert space H which contains the L^2 functions such that their norms on the lefthand side of the estimate in Exercise 3 are finite. Use this estimate to prove that there exists an inverse operator $L^{-1} : L^2 \to H$. Determine the spectrum and the eigenfunctions of L^{-1}.

Hint: This problem requires knowledge of functional analysis. Prove that L defines an injective map with closed range from H to L^2. Use the Lemma of Lax-Milgram to prove that given $f \in L^2$ there is a unique weak solution to $Lu = f$ with $u, \nabla u, xu$ in L^2. Argue that $u \in H$.
Use the Kolmogorov criterion to see that the closure of the unit ball in H is compact in L^2. Thus L^{-1} is a compact operator. We have seen in Exercise Sheet Nr. 2 that Hermite functions are eigenfunctions. Use the compactness to argue that the span of the Hermite functions is dense.