Real and Harmonic Analysis, Problem set 11

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Due on Tuesday, 2016-07-19
Please submit your solutions in groups of two

Problem 1 (Restriction theorem in dimension 2)**.** Let $I \subset \mathbb{R}$ be a compact interval and $\phi : I \to \mathbb{R}$ a C^2 function with $\phi''(x) \geq \lambda > 0$ for all $x \in I$. Let also $F \in L^p(I)$, $1 < p < 4$, and define a measure ν on $\mathbb{R}^2$ by

$$\int f d\nu = \int_I f(x, \phi(x)) F(x) dx.$$

(a) Show that the convolution product measure $\nu * \nu$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$ and find a formula for its density G in terms of F and ϕ.

(b) Show that, for every $1 < r < \infty$,

$$\|G\|_r^r \lesssim \int_{I \times I} |F(t)|^r |F(s)|^r |s-t|^{-(r-1)} dt\, ds$$

(c) Show that $\|G\|_{L^r(\mathbb{R}^2)} \lesssim \|F\|_{L^p(I)}^2$ if $3/r = 2/p + 1$.

(d) Show that $\|\hat{\nu}\|_{L^q(\mathbb{R}^2)} \lesssim \|F\|_{L^p(I)}$ if $q = 3p'$.

Problem 2 (Knapp example)**.** Let σ be the surface measure on the unit sphere in $\mathbb{R}^d$ and let $1 < p, q < \infty$. Suppose that

$$\|f * \sigma\|_{L^{q,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for all} \quad f \in L^p(\mathbb{R}^d).$$

(a) Show that $q \geq p$.

(b) Show that $1/q \geq d/p - (d-1)$.

Suppose now that

$$\|\widehat{f d\sigma}\|_{L^{q,\infty}(\mathbb{R}^d)} \lesssim \|f\|_{L^p(d\sigma)} \quad \text{for all} \quad f \in \mathcal{S}(\mathbb{R}^d).$$

(c) Show that $q \geq \frac{d+1}{d-1} p'$.

Hint: in parts (b) and (c) consider functions f that are (basically) characteristic functions of balls of small radius δ. Estimate the size of the sets on which the function on the left-hand side is almost maximal and let $\delta \to 0$.