Real and Harmonic Analysis, Problem set 10

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Summer term 2016

Due on Tuesday, 2016-07-12
Problems marked as oral will not be graded.
Please submit your solutions in groups of two

Problem 1. Let $\Phi \in C^k[a,b]$ with $k \geq 2$. Assume that $|\Phi^{(k)}| \geq 1$ everywhere on the interval $[a,b]$. Show that

$$\Big| \int_a^b e^{i\lambda\Phi(x)} dx \Big| \leq c_k \lambda^{-1/k}$$

with a constant c_k that does not depend on a, b, Φ.

Problem 2. Consider the polynomial curve $\gamma(t) = (t, t^k)$, $k \geq 2$, and a smooth function with compact support $\psi \in \mathcal{D}(\mathbb{R})$. Define a measure μ by

$$\int_{\mathbb{R}^2} f d\mu = \int_{\mathbb{R}} f(\gamma(t))\psi(t) dt.$$

(a) Show that $|\widehat{\mu}(\xi)| = O(|\xi|^{-1/k})$.

(b) Suppose that $\psi(0) = 0$ and $k \geq 3$. Show that

$$|\widehat{\mu}(0, \xi_2)| = O(|\xi_2|^{-1/(k-1)}).$$

(c) Suppose that $\psi(0) = 0$ and $k = 2$. Show that

$$|\widehat{\mu}(0, \xi_2)| = O(|\xi_2|^{-1} \log |\xi_2|) \text{ for large } \xi_2.$$

(d) Show that the decay rate in part (a) is optimal when $\psi(0) \neq 0$ in the sense that $|\widehat{\mu}(0, \xi_2)| > c|\xi_2|^{-1/k}$ for some $c > 0$ and sufficiently large ξ_2.

Problem 3 (oral)**.** Let P be a real polynomial on $\mathbb{R}^d$ and k a homogeneous function of degree $-d$ that satisfies the cancellation condition $\int_{S^{d-1}} k = 0$.

(a) Show that the limit

$$K(\phi) := \lim_{\epsilon \to 0} \int_{|x|>\epsilon} e^{iP(x)} k(x) \phi(x) dx$$

exists for every Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^d)$ and defines a tempered distribution (the principal value of $e^{iP} k$).

(b) For $d = 1$ show that the Fourier transform $\widehat{K}$ is a bounded function.