Real and Harmonic Analysis, Problem set 7

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Due on Thursday, 2016-06-16
Please submit your solutions in groups of two

This week you are asked to verify some properties of Hermite polynomials that will be useful in next week's lectures (June 14 and 16).

Problem 1 (Diagonalization of the Fourier transform). (a) Show that space of all polynomials is dense in $\mathcal{H} := L^2(\mathbb{R}, e^{-x^2/2}dx)$. Hint: let $f \in \mathcal{H}$ be a function that is orthogonal to all polynomials in this space. Show that the Fourier transform of the function $f(x)e^{-x^2/2}$ vanishes.

(b) The Hermite polynomials are defined by

$$H_m(x) = (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2}.$$

Show the recursive relations $H_{n+1}(x)' = (n+1)H_n(x)$ and $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$. In particular, H_m is a polynomial of degree m with leading coefficient 1, so that the functions H_m form a basis of the space of all polynomials in one variable.

(c) Let $h_m(x) := e^{-x^2/4} H_m(x)$. Show that

$$\langle h_m, h_n \rangle_{L^2(\mathbb{R})} = \langle H_m, H_n \rangle_{\mathcal{H}} = \delta_{mn} \sqrt{2\pi} n!.$$

(d) Prove the generating function identity

$$e^{-t^2/2+xt} = \sum_{m=0}^{\infty} H_m(x) \frac{t^m}{m!}$$

with uniform convergence for x, t in compact subsets of $\mathbb{C}$ and, for fixed $t \in \mathbb{C}$, with convergence in $\mathcal{H}$.

(e) Let $k_m(x) := h_m(\sqrt{4\pi} x)$. Show that these functions are eigenvectors of the Fourier transform: $\widehat{k_m} = (-i)^m k_m$. Hint: consider the Fourier transform of $e^{-t^2/2+xt} e^{-x^2/4}$.

The existence of this diagonalization is not too surprising, because the Fourier transform is a unitary operator whose fourth power is the identity, hence its spectrum is contained in $\{\pm 1, \pm i\}$. By the spectral theorem $L^2(\mathbb{R})$ splits into the orthogonal sum of the corresponding 4 subspaces, on each of which the Fourier transform is a multiple of the identity.