Real and Harmonic Analysis, Problem set 6

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Due on Thursday, 2016-06-09
Problems marked as oral will not be graded.
Please submit your solutions in groups of two

Problem 1. (a) Show that a distribution on $\mathbb{R}^d$ that is homogeneous of degree $-d$ and supported at 0 is a multiple of the Dirac delta distribution (you may use the characterization of distributions with point support).

(b) Let $d \geq 3$ and $\phi(f) = \int_{\mathbb{R}^d} f(x)|x|^{-d+2} dx$, this is a homogeneous distribution of degree $-d+2$. Show that the distribution

$$\sum_{j=1}^{d} \partial_j \partial_j \phi$$

is homogeneous of degree $-d$ and supported at 0.

(c) Let $d = 2$ and $\phi(f) = \int_{\mathbb{R}^d} f(x) \log |x| dx$, this is a tempered distribution. Show that $\phi(f_a) = \phi(f)$ provided that $\int f = 0$. Show that the distribution

$$\sum_{j=1}^{2} \partial_j \partial_j \phi$$

is homogeneous of degree -2 and supported at 0.

(d) Show that the distributions $\sum_{j=1}^{d} \partial_j^2 \phi$ from parts (b) and (c) are in fact non-zero multiples of δ_0 by evaluating them explicitly on radial Schwartz functions of the form $f(x) = g(|x|^2)$ (or $f(x) = g(|x|)$ if you prefer).

Problem 2 (Yukawa potential)**.** (a) Let σ be the surface measure on the unit sphere $S^2 \subset \mathbb{R}^3$. Show that

$$\int e^{-2\pi i \xi \cdot x} d\sigma(x) = \frac{2 \sin(2\pi |\xi|)}{|\xi|}.$$

Hint: write this integral in angular coordinates.

(b) Compute the Fourier transform of the conjugate Poisson kernel $\frac{r}{1+r^2}$ (this is a function in one variable $r \in \mathbb{R}$). Hint: use part (c) of Problem 1 in Problem set 4, where it has been proved that the Fourier transform of the Poisson kernel $\frac{1}{1+r^2}$ equals $\pi e^{-2\pi|\xi|}$.

(c) Show that the Fourier transform of the function $(1 + |x|^2)^{-1}$ on $\mathbb{R}^3$ is a muliple of $e^{-2\pi|\xi|}/|\xi|$. Hint: write the integral defining the Fourier transform in spherical coordinates.

Problem 3 (oral)**.** Let $\psi \in \mathcal{S}(\mathbb{R}^d)^*$ be the distribution given by

$$\psi(f) = \int_{|x| \leq 1} \frac{f(x) - f(0)}{|x|^d} dx + \int_{|x|>1} \frac{f(x)}{|x|^d} dx.$$

Verify that this is indeed a tempered distribution and compute $\psi(f^a)$, where $f^a(x) = a^{-d} f(x/a)$. Deduce from this computation that the Fourier transform $\hat{\psi}$ is not equal to a bounded function in a neighborhood of zero. Use this fact and the characterization of distributions with point support to show that any tempered distribution that agrees with $|x|^{-d}$ away from the origin has unbounded Fourier transform.