Real and Harmonic Analysis, Problem set 5

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Due on Thursday, 2016-06-02
Please submit your solutions in groups of two

Problem 1. (a) Let $\eta \in \mathcal{D}$ be a function such that $\eta \equiv 1$ in a neighborhood of 0 and let $\eta_k(x) = \eta(x/k)$. Show that for every $f \in \mathcal{S}$ we have $\eta_k f \to f$ in $\mathcal{S}$ as $k \to \infty$. In particular, $\mathcal{D}$ is dense in $\mathcal{S}$.

(b) Let $F \in \mathcal{D}^*$ be a distribution. Show that F extends to a tempered distribution if and only if there exist $N \in \mathbb{Z}$ and $A \geq 0$ such that

$$|F(\phi)| \leq AR^N \sup_{|\alpha| \leq N} \sup_x |\partial^\alpha \phi(x)|$$

for all $R \geq 1$ and $\phi \in \mathcal{D}$ supported in $B(0, R)$.

(c) Let $F \in \mathcal{D}(\mathbb{R}^d)^*$ be a homogeneous distribution of degree $\lambda \in \mathbb{R}$, that is, $F_a = a^\lambda F$ for all $a > 0$, where $F_a(\phi) = F(\phi^a)$, $\phi \in \mathcal{D}$, and $\phi^a(x) = a^{-d}\phi(x/a)$. Show that F extends to a tempered distribution.

Problem 2. Let $F \in \mathcal{D}^*$ be a distribution supported in a compact set K. By Problem 1 it extends to a tempered distribution.

(a) For a C^∞ function f on $\mathbb{R}^d$ let $F(f) := F(\eta f)$, where $\eta \in \mathcal{D}$ is a function with $\eta|_K \equiv 1$. Show that this definition does not depend on η and coincides on $\mathcal{S}$ with the continuous extension of F constructed in Problem 1.

(b) Let $f \in \mathcal{S}$ be a Schwartz function. Show that $F * f$, which was defined in class, is also a Schwartz function.

(c) Show that $\widehat{F}$ coincides as a distribution with the function $\xi \mapsto F(e^{-i\xi\cdot})$, where the right-hand side is defined as in part (a). (Hint: insert factors 2π where needed to conform with your favorite normalization of the Fourier transform)