Real and Harmonic Analysis, Problem set 3

Mathematisches Institut
Dr. Diogo Oliveira e Silva
Dr. Pavel Zorin-Kranich
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Due on Thursday, 2016-05-12
Problems marked as oral will not be graded, but will be discussed during the exercise class.
Please submit your solutions in groups of two

Problem 1 (Failure of local L^1 bound for the maximal operator, oral)**.** Let $f(x) = \frac{1_{|x|<1}}{|x|(\log|x|)^2}$. Verify that $f \in L^1(\mathbb{R})$. Show that

$$Mf(x) \geq \frac{c1_{|x|<1}}{|x|\log|x|},$$

and that the latter function is not in L^1_{loc}.

Problem 2 (Whitney cubes, oral)**.** Let $O \subsetneq \mathbb{R}^n$ be an open set. Show that O can be covered by a disjoint collection of cubes Q such that $\operatorname{dist}(Q, \mathbb{R}^n \setminus O) \sim \ell(Q)$, where $\ell(Q)$ denotes the side length. This is the Whitney covering lemma.

Problem 3 (Calderón–Zygmund decomposition)**.** Let $f \in L^1(\mathbb{R}^n)$ and let $\mathcal{Q}$ be a Whitney covering of the set $\{Mf > \lambda\}$. The *Calderón–Zygmund decomposition of f at the level λ* is

$$f = g + b, \qquad b = \sum_{Q\in\mathcal{Q}} b_Q, \qquad b_Q = 1_Q(f - |Q|^{-1}\int_Q f).$$

The function b is called the "bad part" and the function g the "good part" of f.

(a) Show that $g \in L^\infty(\mathbb{R}^n)$.

(b) Show that $\int |b_Q| \leq C_n\lambda|Q|$ and $\int b_Q = 0$ for every $Q \in \mathcal{Q}$, where $|Q|$ denotes the Lebesgue measure of the set Q.

(c) In the case $n = 1$ use the last estimate to bound $\int_{\{Mf\leq\lambda\}} |Hb|$, where H is the Hilbert transform.

(d) Conclude that H has weak type $(1, 1)$.

Problem 4 (Maximally truncated Hilbert transform)**.** It has been proved in the lecture that

$$Hf = \lim_{\epsilon\to 0} H_\epsilon f, \quad H_\epsilon f(x) = \int_{|t|\geq\epsilon} f(x-t)\frac{dt}{t}, \tag{1}$$

for $f \in L^2(\mathbb{R})$ with convergence in $L^2(\mathbb{R})$. The objective of this problem is to show that, in fact, convergence holds almost everywhere.
Let $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$. For a fixed $\epsilon > 0$ write $f = f_1 + f_2$ with $f_1 = f1_{B(x,\epsilon)}$, $f_2 = f1_{B(x,\epsilon)^c}$ ($B(x, \epsilon)$ is the ball of radius ϵ centered at x and $B(x, \epsilon)^c$ is its complement).

(a) Show that $|x - x'| < \epsilon/2$ implies $|Hf_2(x) - Hf_2(x')| \leq CMf(x)$ and conclude

$$|H_\epsilon f(x)| \leq |Hf(x')| + |Hf_1(x')| + CMf(x)$$

for almost every x' such that $|x - x'| < \epsilon/2$.

(b) Estimate the measure of the set $\{x' : |Hf(x')| > \lambda, |x - x'| < \epsilon/2\}$ in terms of MHf and the measure of the set $\{x' : |Hf_1(x')| > \lambda\}$ using the weak type $(1, 1)$ estimate for H. Choose a λ that ensures that these sets do not cover $B(x, \epsilon/2)$ and conclude that the inequality

$$|H_\epsilon f| \leq C(M(Hf) + Mf)$$

holds pointwise.

(c) Show that the operator

$$H_* f(x) := \sup_{\epsilon>0} |H_\epsilon f(x)|$$

is bounded on $L^2(\mathbb{R})$.

(d) Conclude that (1) holds pointwise almost everywhere.