Real and Harmonic Analysis, Problem set 2

Mathematisches Institut
Dr. Diogo Oliveira e Silva
Dr. Pavel Zorin-Kranich
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Due on Thursday, 2016-04-28
Problems marked as oral will not be graded, but will be discussed during the exercise class.
Please submit your solutions in groups of two

Problem 1 (Quasinormed spaces)**.** Show that the functional $\|f\|_p := (\int |f|^p)^{1/p}$, $0 < p < 1$, satisfies the *quasi-triangle inequality*

$$\|f + g\|_p \le C_p(\|f\|_p + \|g\|_p).$$

Consider the space $L^p(\mathbb{R})$ of the functions f with $\|f\|_p < \infty$. Show that there are no non-trivial bounded linear functionals on this space, that is, if $\ell : L^p(\mathbb{R}) \to \mathbb{C}$ is a linear map such that

$$|\ell(f)| \le C\|f\|_p$$

for all $f \in L^p(\mathbb{R})$, then $\ell \equiv 0$. Hint: let $F(x) := \ell(1_{[0,x]})$ and consider $F(x) - F(y)$.

Problem 2 (Weak L^p spaces)**.** The *weak L^p norm* of a function f is the quantity

$$\|f\|_{p,\infty} := \sup_{\lambda>0} \lambda|\{|f| > \lambda\}|^{1/p}, \qquad 0 < p < \infty.$$

(a) Show that $\|f\|_{p,\infty} \le \|f\|_p$ for all $0 < p < \infty$.

(b) Show that the functional $\|f\|_{p,\infty}$ satisfies the quasi-triangle inequality.

(c) Let $0 < r < p < \infty$. Show that the expression

$$\|f\|_{p,\infty,r} := \sup_{0<|E|<\infty} |E|^{-\frac{1}{r}+\frac{1}{p}} (\int_E |f|^r)^{1/r}$$

is equivalent to the weak L^p norm in the sense that there exist constants $0 < c_{p,r} \le C_{p,r} < \infty$ such that $c_{p,r}\|f\|_{p,\infty} \le \|f\|_{p,\infty,r} \le C_{p,r}\|f\|_{p,\infty}$ holds for all f. Hint: in order to show the second inequality use (and prove that it holds if you haven't seen it before) the layer cake representation

$$\int_E |f|^r = r\int_{\lambda=0}^{\infty} \lambda^{r-1}|E \cap \{|f| > \lambda\}|.$$

(d) Suppose that $1 = r < p < \infty$. Show that $\|f\|_{p,\infty,r}$ is a norm (that is, it satisfies the genuine triangle inequality).

(e) Let $0 < p < 1$. Use Problem 1 and the Hahn–Banach theorem to show that $\|f\|_{p,\infty}$ does not admit an equivalent norm.

(f) Show that there is no norm equivalent to $\|f\|_{1,\infty}$. To this end, for each $N \in \mathbb{N}$ construct a sequence of functions f_j, $j = 1, \dots, 2^N$, such that $\|f_j\|_{1,\infty} \le C$ and $\|\sum_{j=1}^{2^N} f_j\|_{1,\infty} \ge cN2^N$.

Problem 3 (Failure of Hasdorff–Young for $p > 2$, oral)**.** Let $\phi : [0, 2\pi] \to \mathbb{C}$ be a smooth function supported on $[0, 1/N]$ and let $\phi_j(x) = \phi(x - j/N)e^{ix\xi_j}$, where $\xi_{j+1} - \xi_j \gg 1$. Compare the growth rates of

$$\|\sum_{j=1}^{N} \phi_j\|_p \text{ and } \|\sum_{j=1}^{N} \hat{\phi}_j\|_{p'}$$

as $N \to \infty$ and conclude that the Hasdorff–Young inequality is false for $p > 2$.

Problem 4 (Logarithmic convexity of L^p norms, oral)**.** Let $0 < p_0 < p < p_1 \le \infty$ and $f \in L^{p_0} \cap L^{p_1}$. Show that for all $0 \le \theta \le 1$ one has

$$\|f\|_{p_\theta} \le \|f\|_{p_0}^{1-\theta}\|f\|_{p_1}^{\theta}, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

using the three lines lemma.