Complex Analysis
Lecture notes

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Contents

1 Fundamentals

A complex number is a pair (x, y) of real numbers. The space $\mathbb{C} = \mathbb{R}^2$ of complex numbers is a two-dimensional $\mathbb{R}$-vector space. It is also a normed space with the norm defined as

$$|(x, y)| = \sqrt{x^2 + y^2}.$$

This is the usual Euclidean norm and induces the structure of a Hilbert space on $\mathbb{C}$. An additional feature that makes $\mathbb{C}$ very special is that it also has a product structure defined as follows (that product is not to be confused with the scalar product of the Hilbert space).

Definition 1.1 (Product of complex numbers)**.** For two complex numbers $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$, their product is defined by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

*Notes by Joris Roos and Gennady Uraltsev.

This defines a map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$. It can be rewritten in terms of another product, the matrix product:

$$(x_1, y_1)(x_2, y_2) = (x_1, y_1) \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix}.$$

In fact, we can embed the complex numbers into the space of real 2×2 matrices via the linear map

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{R}^{2\times 2} \\ (x, y) &\longmapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \end{aligned}$$

The map translates the product of complex numbers into the matrix product. This is very helpful to verify some of the following properties:

1. Commutativity (follows directly from the definition),
2. Associativity,
3. Distributivity,
4. Existence of a unit:

   $$(x_1, y_1) = (1, 0)(x_1, y_1), \text{ and}$$

5. Existence of inverses: if $(x, y) \neq 0$, then

   $$(x, y)\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) = \left(\frac{x^2 + y^2}{x^2 + y^2}, \frac{xy - yx}{x^2 + y^2}\right) = (1, 0).$$

   In terms of the matrix representation this property is based on the fact that non-zero matrices of the form $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ are always invertible:

   $$\det \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x^2 + y^2 \neq 0 \tag{1.1}$$

   for $(x, y) \neq 0$. It also entails that the inverse matrix is again of that form.

Summarizing, the product of complex numbers gives $\mathbb{C}$ the structure of a field. The existence of such a product makes $\mathbb{R}^2$ unique among the higher dimensional Euclidean spaces $\mathbb{R}^d$, $d \geq 2$. Roughly speaking, the reason for this phenomenon is the very special structure of the above 2×2 matrices. In higher dimensions it becomes increasingly difficult to find a matrix representation such that Property 5 is satisfied. The only cases in which it is possible at all give rise to the quaternion ($d = 4$) and octonion ($d = 8$) product, neither of which is commutative (and the latter is not even associative).

Another important property is that we have compatibility of the product with the norm:

$$|(x_1, y_1)(x_2, y_2)| = |(x_1, y_1)| \cdot |(x_2, y_2)|.$$

This is a consequence of the determinant product theorem and the identity

$$|(x, y)| = \sqrt{\det \begin{pmatrix} x & y \\ -y & x \end{pmatrix}}.$$

One consequence of this is that for fixed (x_1, y_1), the map $(x_1, y_1) \mapsto (x_1, y_1)(x_2, y_2)$ is continuous (but of course this can also be derived differently).
We now proceed to introduce the conventional notation for complex numbers.

Definition 1.2. We write $1 = (1, 0)$ to denote the multiplicative unit. $i = (0, 1)$ is called the *imaginary unit*. A complex number (x, y) is written as

$$z = x + iy.$$

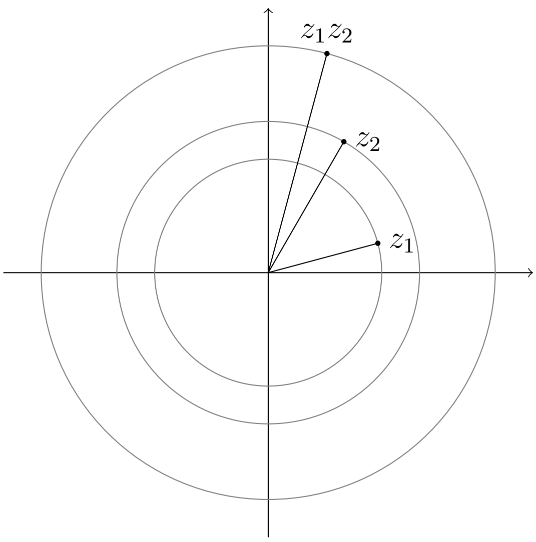
$x =: \mathrm{Re}\,(z)$ is called the *real part* and $y =: \mathrm{Im}\,(z)$ the *imaginary part*. The *complex conjugate* of $z = x + iy$ is given by

$$\overline{z} = x - iy$$

We have the following identities:

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1,$$
$$|z|^2 = z\overline{z} = (x + iy)(x - iy) = x^2 + y^2,$$
$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

The product of complex numbers has a geometric meaning. Observe that the unit circle in the plane consists of those complex numbers z with $|z| = 1$. Say that z_1, z_2 lie on the unit circle. That is, $|z_1| = 1$, $|z_2| = 1$. Then also $|z_1 z_2| = |z_1| \cdot |z_2| = 1$, so also $z_1 z_2$ is on the unit circle. So the linear map $\mathbb{C} \to \mathbb{C}, z_1 \mapsto z_1 z_2$ maps the unit circle to itself. Recall that there are not too many linear maps with this property: only rotations and reflections. Since the determinant is positive by (1.1), it must be a rotation.

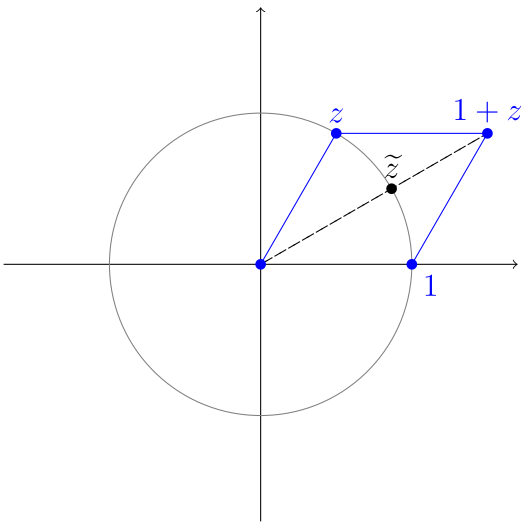


Every non-zero complex number can be written as the product of one on the circle and a real number:

$$z = \frac{z}{|z|}|z|$$

Multiplication with a real number corresponds to stretching, so we conclude from the above that multiplication with a complex number corresponds to a rotation and stretching of the plane.

Example 1.3. We use our recently gained geometric intuition to derive a curious formula for the square root of a complex number. Look at the following picture.



We have given some z with $|z| = 1$ and would like to find $\widetilde{z}$ with $\widetilde{z}^2 = z$. The picture suggests to pick

$$\widetilde{z} = \frac{1+z}{|1+z|}.$$

Indeed we have

$$\widetilde{z}^2 = \frac{(1+z)^2}{(1+z)(1+\overline{z})} = \frac{1+z}{1+\overline{z}} = \frac{z\overline{z}+z}{1+\overline{z}} = z\frac{1+\overline{z}}{1+\overline{z}} = z.$$

Now let $z \neq 0$ be a general complex number and apply the above to $\frac{z}{|z|}$. Then the square roots of z are given by

$$\sqrt{z} = \pm \frac{1+\frac{z}{|z|}}{\left|1+\frac{z}{|z|}\right|}\sqrt{|z|}.$$

We now turn our attention to functions of a complex variable $f : \mathbb{C} \to \mathbb{C}$. A prime example is given by complex power series:

$$\sum_{n=0}^{\infty} a_n z^n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n z^n.$$

To find out when this limit exists we check when the sequence of partial sums is Cauchy. Take $M < N$ and compute:

$$\left|\sum_{n=0}^{N} a_n z^n - \sum_{n=0}^{M} a_n z^n\right| = \left|\sum_{n=M+1}^{N} a_n z^n\right| \leq \sum_{n=M+1}^{N} |a_n z^n| = \sum_{n=M+1}^{N} |a_n|\, r^n,$$

where $r = |z|$. This implies that if $\sum_{n=0}^{\infty} |a_n| r^n$ converges in $\mathbb{R}$, then $\sum_{n=0}^{\infty} a_n z^n$ converges in $\mathbb{C}$. Next, $\sum_{n=0}^{\infty} |a_n| r^n < \infty$ holds if there exists $\widetilde{r} > r$ with $\sup_n |a_n| \widetilde{r}^n < \infty$ because

$$\sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} a_n \widetilde{r}^n \left(\frac{r}{\widetilde{r}}\right)^n \leq \left(\sup_n |a_n| \widetilde{r}^n\right) \sum_{n=0}^{\infty} \left(\frac{r}{\widetilde{r}}\right)^n < \infty.$$

Definition 1.4. The *convergence radius* of a power series $\sum_{n=0}^{\infty} a_n z^n$ is defined as

$$R := \sup\{\widetilde{r} \,:\, \sup_n |a_n| \widetilde{r}^n < \infty\}.$$

- For $z \in D_R(0) = \{z \,:\, |z| < R\}$, the sum $\sum_{n=0}^{\infty} a_n z^n$ converges.
- For $|z| > R$, the sum $\sum_{n=0}^{\infty} a_n z^n$ diverges.

- For $|z| = R$ both convergence and divergence are possible.

Examples 1.5. The *exponential series*

$$e^z := \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

has convergence radius $R = \infty$. The same holds for

$$\cos(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n},$$

$$\sin(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

These combine to give the Euler formula,

$$e^{iz} = \cos(z) + i\sin(z).$$

From Analysis I we know[1] that

$$e^{z_1+z_2} = e^{z_1} e^{z_2}$$

for all $z_1, z_2 \in \mathbb{C}$.

These properties imply that for φ real, $e^{i\varphi}$ lies on the unit circle, in other words that $\sin(\varphi)^2 + \cos(\varphi)^2 = 1$:

$$\sin(\varphi)^2 + \cos(\varphi)^2 = |e^{i\varphi}|^2 = e^{i\varphi}\overline{e^{i\varphi}} = e^{i\varphi}e^{-i\varphi} = e^{i\varphi - i\varphi} = e^0 = 1.$$

Remark 1.6. General polynomials in x, y on $\mathbb{R}^2$ are of the form

$$\sum_{n,m=0}^{N} a_{n,m} x^n y^m = \sum_{n,m=0}^{N} a_{n,m} \left(\frac{z+\overline{z}}{2}\right)^n \left(\frac{z-\overline{z}}{2i}\right)^m = \sum_{n,m=0}^{N} b_{n,m} z^n \overline{z}^m.$$

In complex analysis we only consider the case $b_{n,m} = 0$ for $m \neq 0$.

Definition 1.7. Let $\Omega \subset \mathbb{C}$ be open. A function $f : \Omega \to \mathbb{C}$ is called *complex differentiable* at $z \in \Omega$ if there exists $\delta > 0$ such that $D_\delta(z) := \{w \in \mathbb{C} : |z - w| < \delta\} \subset \Omega$ and the function o, defined by the equation

$$f(z+h) = f(z) + hg(z) + o(h), \tag{1.2}$$

has the property that for all $\varepsilon > 0$ there exists $\delta > 0$ with $|o(h)| < \varepsilon|h|$ for all $|h| < \delta$.

[1] Precisely speaking, we only proved it for real numbers, but the proof is literally the same.

Theorem 1.8. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has convergence radius R, then*

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

also has convergence radius R and for $|z| < R$, f is complex differentiable at z.

Proof. We already know how to differentiate power series from real analysis. The proof of this theorem works exactly the same way as in the real case:

$$\begin{aligned} f(z+h) = \sum_{n=0}^{\infty} a_n (z+h)^n &= \sum_{n=0}^{\infty} \left(a_n z^n + nhz^{n-1} + \sum_{k=2}^{n} a_n \binom{n}{k} h^k z^{n-k} \right) \\ &= f(z) + hg(z) + o(h) \end{aligned}$$

and

$$\left| \frac{o(h)}{h} \right| \le |h| \sum_{n=0}^{\infty} |a_n| n^2 \sum_{k=0}^{n-2} \binom{n+2}{k} |h|^k |z|^{n+2-k} \le |h| \sum_{n=0}^{\infty} |a_n| n^2 (|z| + |h|)^{n+2}.$$

□

Compare this to the real Taylor series in $\mathbb{R}^2$: let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be totally differentiable in z, then there exists a matrix A with

$$f(z+h) = f(z) + Ah + o(h) \tag{1.3}$$

and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|o(h)| \le \varepsilon |h|$ for $|h| < \delta$. Note that the product in (1.3) is the matrix product and the product in (1.2) is the product of complex numbers. They coincide if and only if

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Thus we find that a function $f(z) = (u(x,y), v(x,y))$ that is (real) totally differentiable at z is complex differentiable at z if and only if

$$\frac{\partial u}{\partial x}(z) = \frac{\partial v}{\partial y}(z) \quad \text{and} \quad \frac{\partial u}{\partial y}(z) = -\frac{\partial v}{\partial x}(z). \tag{1.4}$$

These are called the *Cauchy-Riemann differential equations.*

End of lecture 1. April 11, 2016

We will now study some properties of functions on an open complex disk. In particular we will concentrate on the question of regularity and differentiability. In the previous lecture we have mentioned that power series are complex differentiable inside the disk of the radius of convergence. To establish notation let us introduce the following sets.
A We denote by A the set of all power series

$$A := \left\{ \sum_{n=0}^{\infty} a_n z^n \,:\, a_n \in \mathbb{C}, z \in D_R(0) \right\}$$

with radius of convergence at least $R > 0$ so that for all z in the domain $D_R(0) = \{z \,:\, |z - 0| < R\}$ the series converges absolutely (equivalently $sup_n|a_n|\tau^n < \infty$ for any $0 \leq \tau < R$).
As noted previously, A is a subset of the set of all formal power series on $D_R \subset \mathbb{C}$ given by $\sum_{m,n=0}^{\infty} b_{n,m} x^n y^m$ with $x = \mathrm{Re}\,(z)$, $y = \mathrm{Im}\,(z)$. Equivalently these formal series can be expressed as $\sum_{n,m=0}^{\infty} a_{n,m} z^n \bar{z}^m$ and A puts both a restriction on the growth of the coefficients $a_{n,m}$ given by the condition of being convergent on $D_R(0)$ and the additional constraint that $a_{n,m} = 0$ unless $m = 0$.
B We denote by B the set of functions that are complex differentiable in every point of the open disk $D_R(0)$. In particular, as per condition (1.2), B consists of those functions $f : D_R(0) \mapsto \mathbb{C}$ such that for any point $z \in D_R(0)$ and for any increment $h \,:\, |z| + |h| < R$ there exists the complex derivative $g(z) \in \mathbb{C}$ i.e. a complex coefficient such that

$$f(z + h) = f(z) + hg(z) + o(h)$$

where $o(h)$ is some function (depending on z) for which for any $\epsilon > 0$ there exists a $\exists \delta > 0$ such that for any $|h| < \delta$, $|z| + |h| < R$ we have that $o|h| \leq \epsilon |h|$. Recall that this is related to total differentiability on $\mathbb{C} \equiv \mathbb{R}^2$. As a matter one can write the following for a totally differentiable function on $\mathbb{C}$:

$$f(z + h) = f(z) + A(z)h + o(h) \qquad A(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}.$$

Complex differentiability is equivalent to asking the differential as a linear map $A : \mathbb{R}^2 \mapsto \mathbb{R}^2$ can be represented by complex multiplication: $A(z)h = g(z)h$ for some $g(z) \in \mathbb{C}$. This holds if and only if $a(z) = d(z)$ and $b(z) = -c(z)$.
Let us recall the Cauchy-Riemann equations (1.4) and elaborate how they are related to complex differentiability. Setting $f(z) = (u(x, y),\, v(x, y))$, the

equations are given by

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \qquad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}.$$

We can rewrite this equation by defining the following two differential operators called $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ by setting

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Once again setting $f(x,y) = u(x,y) + iv(x,y)$ we can compute

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) = 0$$

It is apparent that the two Cauchy-Riemann equations are just the real and imaginary part of $\frac{\partial f}{\partial \bar{z}}$. Since we have already mentioned that complex differentiability is equivalent to a condition on the differential matrix A that corresponds to the Cauchy-Riemann equations in terms of partial derivatives, it follows that a function is complex differentiable if and only if it is totally differentiable and has $\frac{\partial f}{\partial \bar{z}} = 0$. Furthermore, if f is complex differentiable then we write

$$f'(z) := \frac{\partial}{\partial z} f(z).$$

Finally, in terms of the the real and imaginary part separately we have

$$\frac{\partial}{\partial z} f(z) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right).$$

C We denote by C the subset of continuous functions $f : D_R(0) \mapsto \mathbb{C}$ such that the following integral condition holds

$$\int_{(a,b,c)} f(z)dz = 0 \qquad \forall a,b,c \in D_R(0).$$

Here (a,b,c) is the (oriented) boundary of the (oriented) triangle, also referred to as a simplex, formed by the points a, b, and c. We will identify (a,b,c) by the closed path composed of the three segments $a \to b \to c \to a$. The above integral is a special case of an integral along a path of a complex function. For now we restrict ourselves to the case were the support of the path is a complex segment, parametrized in linear fashion.

Definition 1.9 (Integral of a complex function along a segment)**.** Consider the segment (a, b) with $a, b \in \mathbb{C}$ and a complex-valued continuous function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ defined on an open neighborhood of (a, b). We set the integral of the function f along (a, b) to be

$$\int_{(a,b)} f(z)dz := \int_0^1 f\left(bt + a(1-t)\right)(b-a)dt.$$

Here the integrand on the right hand side is a function $[0, 1] \mapsto \mathbb{C} \equiv \mathbb{R}^2$ and the integral is simply calculated coordinate-wise. Notice however that the integrand itself $f\left(bt + a(1-t)\right) \cdot (b-a)$ is expressed itself as a *complex* product.

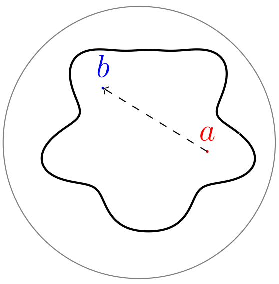


Figure 1: A segment defining a path from a to b.

This definition of the integral over a segment corresponds to the well known concept of a path integral, and extends it to complex functions:

$$\int_{(a,b)} f(z)dz = \int_0^1 f\left(bt + a(1-t)\right)(b-a)dt = \int_\gamma f d\gamma = \int_0^1 f(\gamma(t))\gamma'(t)dt$$

with $\gamma(t) = bt + a(1-t)$ as the path that parameterizes the segment. We naturally extend this definition to the three oriented segments of the boundary of a triangle by setting

$$\int_{(a,b,c)} f(z)dz := \int_{(a,b)} f(z)dz + \int_{(b,c)} f(z)dz + \int_{(c,a)} f(z)dz.$$

Finally notice that the definition of integrating along a path is oriented and as such we have

$$\int_{(a,b)} f(z)dz = -\int_{(b,a)} f(z)dz.$$

This can be easily verified by a change of variables.

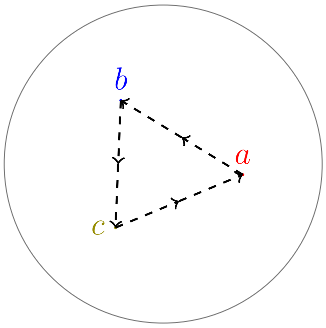


Figure 2: A triangle and its oriented boundary.

The characterization of the set C in terms of path integrals is geometric and does not rely on the smoothness of f. As a matter of fact we require f merely to be continuous. However we will now see that integral over triangles condition over all possible triangles implies stronger structure results and in particular that f is actually smooth and complex differentiable.

Theorem 1.10. *The classes of functions we introduced coincide i.e.* $A = B = C$.

$\mathbf{A} \subset \mathbf{B}$ The rules of differentiation of power series imply immediately the inclusion $A \subset B$.
$\mathbf{A} \subset \mathbf{C}$ We will now show directly that the path integral of a power series along a closed path, and specifically (a, b, c) is zero. In previous courses of analysis we have seen a similar statement for gradient fields and the proof followed from the existence of a primitive. We can, however, deduce the existance of a primitive of a power series formally and this will provide us with the needed elements to adapt a similar approach.
Recall the definition of the set A: $f \in A$ is of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let us define its primitive via

$$F(z) := \sum_{n=0}^{\infty} \frac{1}{n+1} a_n z^{n+1}.$$

Clearly $F \in A$ since it is a power series and its radius of convergence is not smaller than that of f. This follows simply from the bound on the n^{th} coefficient of F by that of f:

$$\frac{1}{n+1}|a_n| \leq |a_n|.$$

We claim that F is effectively a primitive of f and in particular

$$\int_{(a,b)} f(z)dz = \int_0^1 f\left(a(1-t) + bt\right)(b-a)dt = F(b) - F(a).$$

The first equality is just the definition of a complex path integral. To show the second equality let us define $g(y) = F(\gamma(t))$ with $\gamma(t) = a(1-t)+bt$ and let us show that

$$g'(t) = f\left(a(1-t)+bt\right)(b-a).$$

This is essentially the chain rule for complex-valued complex differentiable functions. We write

$$\begin{aligned} g(t+h) &= F(\gamma(t+h)) = F\left(\gamma(t)+(b-a)h\right) \\ &= F(\gamma(t)) + (b-a)hf\left(\gamma(t)\right) + o\left((b-a)h\right) \\ &= g(t) + (b-a)hf\left(\gamma(t)\right) + o\left((b-a)h\right) \end{aligned}$$

Here we used that the complex differential of F in $\gamma(t)$ is given by $f\left(\gamma(t)\right)$ and that $(b-a)h$ is a small complex increment. Notice also that h is a real increment. We have thus that

$$\int_0^1 f\left(a(1-t)+bt\right)(b-a)dt = F(b) - F(a)$$

and

$$\begin{aligned} \int_{(a,b,c)} f(z)dz =& \int_{(a,b)} f(z)dz + \int_{(b,c)} f(z)dz + \int_{(c,a)} f(z)dz \\ =& F(b) - F(a) + F(c) - F(b) + F(a) - F(c) = 0 \end{aligned}$$

B $\subset$ **C** This statement is known as "Theorem of Goursat". Let $f \in B$ be complex differentiable in $D_R(0)$. We must show that for any $\widetilde{r} < R$ and $\forall a,b,c \in \overline{D_{\widetilde{r}}(0)}$ one has $\int_{(a,b,c)} f(z)dz = 0$. It is sufficient to show that for any $\epsilon > 0$ and $\forall a,b,c \in D_{\widetilde{r}}(0)$ we have that

$$\left|\int_{(a,b,c)} f(z)dz\right| \leq \epsilon \max\left(|b-a|, |c-b|, |a-c|\right)^2.$$

The argument we present relies on an induction on scales. The term

$$\max\left(|b-a|, |c-b|, |a-c|\right)^2$$

on the right hand side of the above entry is a measure of the scale of "how large" or the *scale* of the triangle. We will show that the statement holds for triangles that have sufficiently small scale and then to an induction argument that will show that is the statement holds for a certain scale it also holds for triangle up to twice as large. This would allow us to conclude the statement for all triangles.

Part 1: We start by showing that the above bound holds for all points $a, b, c \in D_{\tilde{r}}(0)$ with $\max(|b-a|, |c-b|, |a-c|) < \delta_{min}$ for some $\delta_{min} > 0$. For any $z \in D_{\tilde{r}}(0)$ there exists $\delta = \delta(z)$ such that $\forall |h| < \delta$ we have

$$f(z+h) = f(z) + hf'(z) + o(h) \qquad \text{with } |o(h)| < \frac{\epsilon}{8}|h|.$$

Reasoning by compactness we can find a finite set $z_1, \ldots, z_N$ such that $\overline{D_{\tilde{r}}(0)} \subset \bigcup_{j-1}^{N} D_{\delta(z_i)/3}(z_i)$ where $\delta(z_i)$ is the radius for which the above bound holds. Setting $\delta_{min} := \frac{\min_i \delta(z_i)}{3}$ one has that $\forall z \in \overline{D_{\tilde{r}}(0)}$ $\forall |h| < \delta_{min}$ we have via the triangle inequality

$$f(z+h) = f(z) + hf'(z) + o(h) \qquad \text{with } |o(h)| < \frac{\epsilon}{4}|h|.$$

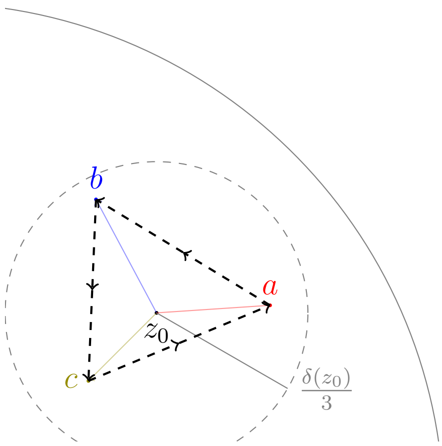


Figure 3: A triangle in a small circle

Now consider two point a, b with $|b-a| < \delta_{min}$. We can evaluate the contribution of the three terms of the expansion of f to the line integral.

$$\int_{(a,b)} f(z)dz = \int_{(a,b)} f(z_0) + (z-z_0)f'(z_0) + o(z-z_0)dz$$

The first term gives

$$\int_0^1 f(z_0)(b-a)dt = f(z_0)(b-a).$$

The second term gives

$$\begin{aligned}
&\int_0^1 f'(z_0)\,(bt - a(1-t) - z_0)\,(b-a)dt \\
=&f'(z_0)\left(\frac{1}{2}(b+a)(b-a) - a(b-a) - z_0(b-a)\right) \\
=&f'(z_0)\left(\frac{1}{2}(b^2-a^2) + z_0(b-a)\right)
\end{aligned}$$

We have crucially used complex differentiability of f here. As a matter of fact the algebraic manipulation relied on the commutativity of complex multiplication. If f were just any totally differentiable function then $f'(z_0)$ would be substituted by some arbitrary 2×2 matrix and the above identity would not necessarily hold.
Summing up the contributions of the three terms we obtain

$$\begin{aligned}
&\int_{(a,b,c)} f(z)dz = \int_{(a,b)} f(z)dz + \int_{(b,c)} f(z)dz + \int_{(c,a)} f(z)dz = \\
&= f(z_0)(b-a+c-b+a-c) \\
&+ f'(z_0)\left(\frac{1}{2}(b^2-a^2+c^2-b^2+a^2-c^2) + z_0(b-a+c-b+a-c)\right) \\
&+ \int_{(a,b,c)} o(z-z_0)dz
\end{aligned}$$

All terms except the last vanish while for the last we have the bound

$$\begin{aligned}
&\left|\int_{(a,b,c)} f(z)dz\right| = \left|\int_{(a,b,c)} o(z-z_0)dz\right| \\
&< \frac{\epsilon}{4}\left(|b-a| + |c-b| + |a-c|\right) \max_{z\in(a,b,c)} |z-z_0| \\
&< \frac{3\epsilon}{4} \max\left(|b-a|, |c-b|, |a-c|\right)^2
\end{aligned}$$

as required.
Part 2: We have now proved that the bound we seek holds for triangles that are small enough. In particular we require that $\max(|b-a|, |c-b|, |a-c|) < \delta_{min}$. We will now show an inductive procedure that shows that if the statement holds for when $\max(|b-a|, |c-b|, |a-c|) < \delta$ then the same is true if $\max(|b-a|, |c-b|, |a-c|) < 2\delta$.
The main idea is given by decomposing a triangle into smaller triangles in a uniform way.

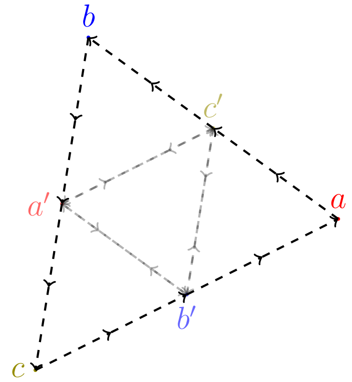


Figure 4: Decomposing triangles into smaller ones

To do so we use the median points as shown in figure 4. Let a', b', c' be the median points of the sides of (a, b, c) opposite of the respective vertices. We have

$$\begin{aligned}\int_{(a,b,c)} f(z)dz &= \int_{(a,c',b')} f(z)dz + \int_{(b,a',c')} f(z)dz + \int_{(c,b',a')} f(z)dz \\ &\quad + \int_{(a',b',c')} f(z)dz \\ \left|\int_{(a,b,c)} f(z)dz\right| &< \epsilon\Big(\max\left(|c'-a|, |b'-c'|, |a-b'|\right)^2 + \max\left(...\right)^2 + \max\left(...\right)^2 \\ &\quad + \max\left(|b'-a'|, |c'-b'|, |a'-c'|\right)^2\Big) \\ &< 4\epsilon\frac{\max\left(|b-a|, |c-b|, |a-c|\right)^2}{4}\end{aligned}$$

as required. The crucial observation is that once we divide by the medians we obtain four triangles for which the largest of side lengths is bounded by a small (1/2) factor of the lengths of the original triangle. This implies that first of all we may apply the assumptions at previous scale and that we obtain a bound with the same constant.

End of lecture 2. April 14, 2016

We will prove the following stronger version of Goursat's theorem.

Theorem 1.11. *Let $z_0 \in D_R(0)$, $f : D_R(0) \to \mathbb{C}$ continuous and complex differentiable at all points of $D_R(0)\backslash\{z_0\}$. Then $f \in C$.*

Proof. It suffices to show that for all $\widetilde{r} < R$, $a, b, c \in D_{\widetilde{r}}(0)$ we have

$$\int_{(a,b,c)} f(z)dz = 0.$$

Let $10\delta = R - \widetilde{r}$. By the same argument as in the proof of Goursat's theorem it suffices to show this for small triangles: for all $a, b, c \in \overline{D_{\widetilde{r}}(0)}$ with $\max(|a-b|, |b-c|, |c-a|) \leq \delta/10$.

Case 1. $z_0 \notin D_{\delta/3}(a)$. Then $\int_{(a,b,c)} f(z)dz = 0$ holds by Goursat's theorem.

Case 2. $z_0 \in D_{\delta/3}(a)$. It suffices to show $\int_{(a,b,z_0)} f(z)dz = 0$ because

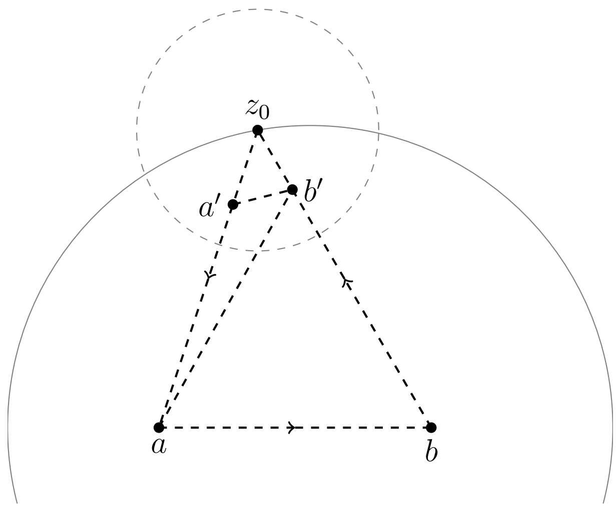
$$\int_{(a,b,c)} = \int_{(a,b,z_0)} + \int_{(b,c,z_0)} + \int_{(c,a,z_0)}.$$

We can also assume that the angle at z_0 is acute (if it is not acute, we bisect the angle at z_0 and consider the two resulting triangles). Next, construct a circle through z_0 that contains (a, b, z_0). We can do this such that the radius is at most δ.

Let $\varepsilon > 0$ be arbitrary. We will show

$$\left| \int_{(a,b,z_0)} f(z)dz \right| \leq \varepsilon.$$

By continuity of f at z_0 we can choose points a' on (a, z_0) and b' on (b, z_0) such that $|f(z) - f(z_0)| < \varepsilon/(3\delta)$ for all z on the triangle (a', b', z_0).



By Goursat's theorem we have

$$\int_{(a,b,b')} f(z)dz = \int_{(a',a,b')} f(z)dz = 0$$

so that

$$\int_{(a,b,z_0)} f(z)dz = \int_{(a',b',z_0)} f(z)dz.$$

We estimate,

$$\begin{aligned}\left|\int_{(a',b',z_0)} f(z)dz\right| &= \left|\int_{(a',b',z_0)} f(z) - f(z_0)dz\right| \\ &\leq \underbrace{\int_0^1 |f(b't + a'(1-t)) - f(z_0)||b' - a'|dt + \cdots}_{<\varepsilon}\end{aligned}$$

□

As a precursor to showing $B \subset A$ we first prove the following.

Theorem 1.12. *Let $f : D_R(0) \to \mathbb{C}$ complex differentiable on $D_R(0)$. Then for all $z_1 \in D_R(0)$ there exists $\delta > 0$ such that f can be represented by a convergent power series on $D_\delta(z_0) \subset D_R(0)$.*

Remark 1.13. In particular, this entails that functions which are complex differentiable in a neighborhood are automatically infinitely often complex differentiable.

This is a consequence of what is called *Cauchy's integral*.

Proof. For $w \in D_R(0)$ we consider the function

$$g_w(z) = \frac{f(z) - f(w)}{z - w}$$

with the understanding that $g_w(w) = f'(w)$. This function is continuous on $D_R(0)$ and complex differentiable on $D_R(0)\backslash\{w\}$. Continuity of g_w in w is a consequence of complex differentiability of f in w. Complex differentiability of g_w in $D_R(0)\backslash\{w\}$ follows by the product rule since $f(z) - f(w)$ and $\frac{1}{z-w}$ are both complex differentiable. Let us show the complex differentiability of $\frac{1}{z}$ on $\mathbb{C}\backslash\{0\}$ directly from the definition:

$$\frac{1}{z+h} - \frac{1}{z} = \frac{z-(z+h)}{z(z+h)} = \frac{-h}{z^2} + \frac{h}{z^2} - \frac{h}{z(z+h)} = -\frac{h}{z^2} + \frac{h^2}{z^2(z+h)} = \frac{-h}{z^2} + o(h)$$

where $o(h) = h^2/(z^2(z+h))$ so that

$$|o(h)| \leq |h^2| \left|\frac{1}{z^2(z+h)}\right| \leq |h|^2 \left|\frac{2}{z^3}\right|.$$

provided that $|h| < \frac{|z|}{2}$.

Choose $a, b, c \in D_R(0)$ such that z_0 lies in the interior of the triangle (a, b, c). Further, pick $\delta > 0$ small enough so that the circle of radius 2δ around z_0 is contained in the interior of the triangle (a, b, c).
Theorem 1.11 yields

$$\int_{(a,b,c)} g_w(z)dz = 0$$

for all $w \in D_\delta(z_0)$. That is,

$$\int_{(a,b,c)} \frac{f(z)}{z-w}dz = \left(\int_{(a,b,c)} \frac{dz}{z-w}\right) f(w)$$

Our claim is that

$$\int_{(a,b,c)} \frac{dz}{z-w} = \pm 2\pi i, \tag{1.5}$$

where the sign is according to whether the triangle (a, b, c) is oriented counter-clockwise $(+)$ or clockwise $(-)$. For the remainder of this proof, let us assume it is oriented counter-clockwise. We defer the proof of this claim to the end and first show how to use the equality

$$f(w) = \frac{1}{2\pi i} \int_{(a,b,c)} \frac{f(z)}{z-w}dz$$

to develop f into a convergent power series. The crucial point here is that on the right hand side, the free variable w no longer occurs inside the argument of f. Therefore we just need to know how to develop $w \mapsto \frac{1}{z-w}$ into a power series around z_0:

$$\frac{1}{z-w} = \frac{1}{(z-z_0)(w-z_0)} = \frac{1}{z-z_0} \cdot \frac{1}{1-\frac{w-z_0}{z-z_0}} = \frac{1}{z-z_0} \sum_{n=0}^{\infty} \left(\frac{w-z_0}{z-z_0}\right)^n.$$

As a consequence,

$$\begin{aligned}\int_{(a,b,c)} \frac{f(z)}{z-w}dz &= \int_{(a,b,c)} \frac{1}{z-z_0} f(z) \sum_{n=0}^{\infty} \left(\frac{w-z_0}{z-z_0}\right)^n dz \\ &= \sum_{n=0}^{\infty} \left(\int_{(a,b,c)} \frac{f(z)}{(z-z_0)^{n+1}}dz\right)(w-z_0)^n dz,\end{aligned}$$

where the interchange of integration and summation is justified by uniform convergence of the power series since $2|w-z_0| < 2\delta < |z-z_0|$ by construction.

It remains to prove (1.5). For starters we calculate

$$\int_{(a,b)} \frac{1}{z-w} dz = \int_0^1 \frac{b-a}{(b-a)t+a-w} dt = \int_0^1 \frac{1}{t+\frac{a-w}{b-a}} dt.$$

Temporarily denote $\frac{a-w}{b-a} = x + iy$ with x, y real numbers. Decompose the integral into real and imaginary part:

$$\int_0^1 \frac{1}{t+x+iy} dt = \int_0^1 \frac{(t+x)-iy}{(t+x)^2+y^2} dt = \int_0^1 \frac{t+x}{(t+x)^2+y^2} dt + i \int_0^1 \frac{-y}{(t+x)^2+y^2} dt.$$

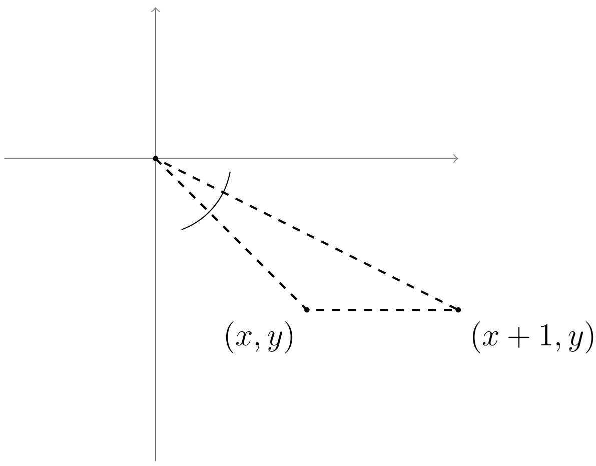
Now we are only dealing with two real integrals that we can evaluate. The first equals

$$\frac{1}{2}\int_x^{x+1} \frac{2t}{t^2+y^2} dt = \frac{1}{2}\left(\log((x+1)^2+y^2) - \log(x^2+y^2)\right) = \log \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{x^2+y^2}}. \tag{1.6}$$

The second equals

$$-\int_x^{x+1} \frac{y}{t^2+y^2} dt = -\int_{x/y}^{(x+1)/y} \frac{1}{s^2+1} ds = -\arctan\left(\frac{x+1}{y}\right) + \arctan\left(\frac{x}{y}\right). \tag{1.7}$$

□



The angle at 0 in the triangle $(0, x+iy, x+1+iy)$ equals $\pm$(1.7). Since addition and multiplication with complex numbers preserves angles, that angle equals the angle at w in the triangle (w, a, b) (the two triangles are similar).

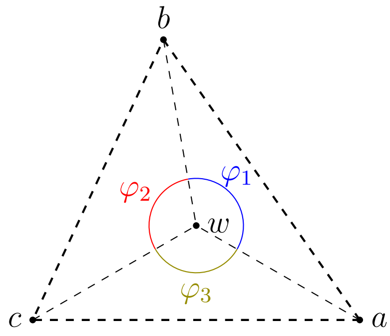
For the same reason we have

$$\log \frac{|(x+1,y)|}{|(x,y)|} = \log \frac{|b-w|}{|a-w|}.$$

Applying the same reasoning to the other two segments (b,c), (c,a) we get

$$\int_{(a,b,c)} \frac{1}{z-w} dz = \overbrace{\log\left(\frac{|b-w|}{|a-w|}\frac{|c-w|}{|b-w|}\frac{|a-w|}{|c-w|}\right)}^{=0} + i(\varphi_1+\varphi_2+\varphi_3) = 2\pi i.$$

The last equality is by inspection of the figure:



End of lecture 3. April 18, 2016

Let us recall the classes of complex-valued functions on the disk $D_R(0)$ that we have introduced so far.

$$A := \left\{ \sum_{n=0}^{\infty} a_n z^n \colon \text{the series converges absolutely on } D_R(0) \right\}$$
$$B := \{ f : D_R(0) \mapsto \mathbb{C} \colon f \text{ is complex differentiable } \forall z \in D_R(0) \}$$
$$C := \left\{ f : D_R(0) \mapsto \mathbb{C} \colon f \in C(D;\mathbb{C}),\ \int_{(a,b,c)} f(z)dz = 0\ \forall a,b,c \in D_R(0) \right\}$$

Additionally we have also introduced a new class $\tilde{A}$ of functions that are locally power series:

$$\tilde{A} := \left\{ f : D_R(0) \mapsto \mathbb{C} \colon f(z_0+h) = \sum_{n=0}^{\infty} a_n(z_0)h^n\ \forall z_0 \in D_R(0)\ |h| < \delta_{z_0} \right\}.$$

where the local power series converges representation converges absolutely for on a disk $D_{\delta_{z_0}}(z_0)$.
We have already seen $A \subset B$, $A \subset C$, $B \subset C$.We will now pass to showing the inclusion $C \subset B$ and we will then conclude that $B \subset A$. It has already been shown that $B \subset \tilde{A}$ via an imporved Goursat's theorem.

Proposition 1.14 (Morera's Theorem: $C \subset B$)**.** *Let $f : D_R(0) \mapsto \mathbb{C}$ be a continuous function such that for any three point $a, b, c \in D_R(0)$ one has*

$$\int_{(a,b,c)} f(z)dz = 0.$$

Set $F(z_1) := \int_{(0,z_1)} f(z)dz$ for any point $z_1 \in D_R(0)$. Then F is complex differentiable in any point z and

$$F(z_1 + h) = F(z_1) + \int_{(z_1,z_1+h)} f(z)dz$$

if $h \in \mathbb{C}$ is such that $z_1 + h \in D_R(0)$.

Proof. Clearly the contour integral condition applied to the triangle of the points $(0, z_1 + h, z_1)$ gives

$$\begin{aligned} F(z_1 + h) &= \int_{(0,z_1+h)} f(z)dz \\ &= \int_{(0,z_1+h,z_1)} f(z)dz + \int_{(0,z_1)} f(z)dz + \int_{(z_1,z_1+h)} f(z)dz \\ &= F(z_1) + \int_{(z_1,z_1+h)} f(z)dz. \end{aligned}$$

To obtain complex differentiability we estimate

$$\begin{aligned} F(z_1 + h) &= F(z_1) + \overbrace{\int_{(z_1,z_1+h)} f(z_1)dz}^{f(z_1)h} + \int_{(z_1,z_1+h)} (f(z) - f(z_1))\, dz \\ &= F(z_1) + f(z_1)h + \int_0^1 (f(z_1 + ht) - f(z_1))\, hdt = F(z_1) + f(z_1)h + o(h) \end{aligned}$$

with $o(h)$ such that for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|o(h)| \leq \int_0^1 |f(z_1 + ht) - f(z_1)|\, |h|dt \leq \epsilon|h|$ if $|h| < \delta$. The last inequality follows from the continuity of f.

We already shown that $B \subset \tilde{A}$. Applying this to F shows that it is locally a power series. In the above expression we have shown that $f = F'$ and thus $f = F'$ formally as power series and it converges absolutely at least the same radius on which F converges and thus $f \in B$. $\square$

We now prove the inclusion $B \subset A$. To do so we need a "global" argument. The local argument gives $B \subset \tilde{A}$. We need to show that for any radius $R' < R$ (and in particular we will need to choose radii $R' < R'' < R''' < R$) the power series representing $f \in \tilde{A}$ in 0 actually converges on $D_{R'}(0)$.

Remark 1.15. Notice that the power series of $f \in \tilde{A}$ can be obtained in any given point (in this case in 0) using the Taylor expansion

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) z^n.$$

The identity can be checked by deriving both sides n times and evaluating the expression in 0.

Figure 5: Discretization of an integral along a cirlce

For any fixed point $z_1 \in D_{R'}(0)$ the function $z \mapsto \frac{f(z)-f(z_1)}{z-z_1}$ is complex differentiable at any point $z \in D_{R'}(0) \setminus \{z_1\}$. This is strait-forward by applying the chain rule to the composition and product of continuous, complex-differentiable functions $f(z) - f(z_1)$ and $\frac{1}{z-z_1}$. Furthermore $z \mapsto f(z) - f(z_1)$ is complex differentiable in z_1 so

$$f(z) - f(z_1) = f'(z_1)(z - z_1) + o(z - z_1).$$

This implies that $\frac{f(z)-f(z_1)}{z-z_1}$ is continuous in z_1 and the value in z_1 is precisely $f'(z_1)$. Let us choose a sequence of 2^n points $(a_1, \dots, a_{2^n})$ on the circle $\{z \in \mathbb{C} : |z| = R'''\}$ going counterclockwise so that the segements (a_{i-1}, a_i) lie in $\overline{D_{R'''}(0)} \setminus D_{R''}(0)$. For example just set $a_j := R''' e^{i2\pi 2^{-n} j}$. Using the extention of Goursat's theorem 1.11 we know that all contour integrals of f over the triangles vanish (a_{i-1}, a_i, z_1) so we can write

$$0 = \sum_{i=1}^{2^n} \int_{(a_{i-1}, a_i, z_1)} \frac{f(z) - f(z_1)}{z - z_1} dz = \sum_{i=1}^{2^n} \int_{(a_{i-1}, a_i)} \frac{f(z) - f(z_1)}{z - z_1} dz$$

where the second equality holds because the radial segments of the integral cancel out. Thus we have

$$\sum_{i=1}^{2^n} \int_{(a_{i-1},a_i)} \frac{f(z)}{z-z_1} dz = \sum_{i=1}^{2^n} \int_{(a_{i-1},a_i)} \frac{f(z_1)}{z-z_1} dz$$
$$= f(z_1) \left(\sum_{i=1}^{2^n} \ln \frac{|a_i - z_1|}{|a_{i-1} - z_1|} + i(\phi_i - \phi_{i-1}) \right)$$
$$= f(z_1) 2\pi i$$

where ϕ_i is the argument of $a_i - z_1$. This follows from computations in (1.6) and (1.7). On the other have the above expression also equals to

$$\sum_{i=1}^{2^n} \int_{(a_{i-1},a_i)} \frac{f(z)}{z} \sum_{m=0}^{\infty} \left(\frac{z_1}{z} \right)^m dz = \sum_{m=0}^{\infty} z_1^m \sum_{i=1}^{2^n} \int_{(a_{i-1},a_i)} \frac{f(z)}{z^{m+1}} dz \tag{1.8}$$

This converges uniformly when $|z_1| < R'$ and $|z| > R''$. Notice that $f(z)$ for $z \in D_{R'''}(0)$ is uniformly bounded and $|z^{-m}| < (R''')^{-m}$ so for each integral we have the bound

$$\left| \int_{(a_{i-1},a_i)} \frac{f(z)}{z^{m+1}} dz \right| \le \|f \mathbf{1}_{D_{R'''}(0)}\|_{sup} (R''')^{-m-1} |a_i - a_{i-1}|.$$

Finally since via geometrical considerations we have that $\sum_{i=1}^{2^n} |a_i - a_{i-1}| \le 2\pi R'''$ by we have that each coefficient satisfies the bound

$$\left| \sum_{i=1}^{2^n} \int_{(a_{i-1},a_i)} \frac{f(z)}{z^m} dz \right| < 2\pi (R''')^{-m} \|f \mathbf{1}_{D_{R'''}(0)}\|_{sup}.$$

This implies that the series (1.8) has a convergence radius given at least by R'''.

Finally we remark a nice formula for the contour integral of $\frac{1}{z}$ over the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Notice that the function $\frac{1}{z}$ does not fall into the class of functions we have defined complex countour integrals for. As a matter of fact $\frac{1}{z}$ is defined on the punctured disk $D_R(0) \setminus \{0\}$ for any $R > 0$ and is complex differentiable in any point where it is defined. However $\frac{1}{z}$ is not even continuous in $z = 0$ and as such none of the above theorems apply to it in the standard form. In particular we have seen that the integral over a triangle (a, b, c) containing 0 of $\frac{1}{z}$ is non-zero and equal to $2\pi i$ if it is counterclockwise (positive) oriented.

However we can define the path integral over a sufficiently smooth path $\gamma : [a, b] \mapsto \mathbb{C}$ by setting

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Here we intend that that the parametrization of the circle is counterclockwise and is given by $\gamma : t \in [0, 2\pi) \mapsto e^{it} \in S^1$ so that

$$\int_{S^1} \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{\gamma(t)}\gamma'(t)dt = \int_0^{2\pi} \frac{1}{e^{it}}ie^{it}dt = 2\pi i.$$

This discussion justifies spending some time on definig path integrals for complex functions and highlighting the important aspects of path integrals of complex differentiable functions specifically.

Definition 1.16 (Non self-intersecting curve C in $\mathbb{C}$)**.** A non self-intersecting curve C in $\mathbb{C}$ is the graph of an injective continuous path $\gamma : [a, b] \mapsto \mathbb{C}$.

Note that if C is a curve that is a graph of $\gamma : [a, b] \mapsto C \subset \mathbb{C}$ then γ is bijective $\gamma^{-1} : C \mapsto [a, b]$ is also continuous.
We can see this by reasoning by contradiction. Clearly the inverse $\gamma^{-1} : \mathbb{C} \mapsto [a, b]$ is defined pointwise because of the injectivity of γ. Suppose that the inverse γ^{-1} is not continuous. That means that there exist two sequences of $(t_n), (\tilde{t}_n)$ such that

$$\liminf_{n\to\infty} |t_n - \tilde{t}_n| = \epsilon > 0 \qquad \lim_{n\to\infty} |\gamma(t_n) - \gamma(\tilde{t}_n)| = 0.$$

Since the interval $[a, b]$ is compact we can restrict ourselves to a subsequence such that

$$\lim_{n\to\infty} t_n = t \in [a, b] \qquad \lim_{n\to\infty} \tilde{t}_n = \tilde{t} \in [a, b] \qquad \liminf_{n\to\infty} |t_n - \tilde{t}_n| > \epsilon.$$

Thus $|t - \tilde{t}| > \epsilon$ but by the continuity of γ we have that $\lim_{n\to\infty} \gamma(t_n) = \lim_{n\to\infty} \gamma(\tilde{t}_n) = \gamma(t) = \gamma(\tilde{t})$. This contradicts injectivity since $t \neq \tilde{t}$.
Suppose now that two paths $\gamma_1 : [a_1, b_1] \mapsto \mathbb{C}$ and $\gamma_1 : [a_2, b_2] \mapsto \mathbb{C}$ have the same image C and suppose that C is continuous and non self-intersecting. Then $\gamma^{-1}\gamma_2 : [a_2, b_2] \mapsto [a_1, b_1]$ is a continous bijection with continuous inverse. The domain of this function and its image are real intervals, thus the function must be monotone and the image of $\{a_2, b_2\}$ must be $\{a_1, b_1\}$. We can thus define the direction of parameterization by asking that γ_1 and γ_2 paramtetrize $\mathbb{C}$ in the same direction if $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$.
We can identify a directed non self-intersecting graph $C \subset \mathbb{C}$ by the family of

all paths that parametrize C in the same direction. This procedure induces a well defined order on C by imposing

$$\gamma(t_1) < \gamma(t_2) \iff t_1 < t_2.$$

Furthermore an odering, together with the fact that C is in (ordered) bijection with a closed real interval shows that sup, inf exists of any subset of C and lim sup lim inf of a sequence $z_n \in C$ is also well defined. Actually to be able to define these notions we do not need C the parametrization.
With such a notion of ordering we can define a non-intersecting curve C to be rectifiable if

$$\sup_{\substack{n, z_0 < \cdots < z_n \\ z_0, \ldots z_n \in C}} \sum_{i=1}^{n} |z_i - z_{i-1}| < \infty.$$

Its arc-length parametrization is then given by introducting the function

$$\beta : C \mapsto [0, L] \qquad \beta(z) = \sup_{\substack{n, z_0 < \cdots < z_n < Z \\ z_0, \ldots z_n \in C}} \sum_{i=1}^{n} |z_i - z_{i-1}|.$$

We leave the following as an exercise

Exercise 1.17 (Arc Length Parametrization). *β^{-1} is a parametrization of C by a segment $[0, L]$ and β^{-1} is 1-Lipschitz i.e. $|\beta^{-1}(t_2) - \beta^{-2}(t_2)| \leq |t_2 - t_1|$. We call L the length of the curve*

For rectifiable curves the concept of path integrals is natural

Definition 1.18 (Path integral).

$$\int_C f(z)dz := \lim_{\epsilon \to 0} \sum_{\substack{a = z_0 < \cdots < z_n = b \\ |z_i - z_{i-1}| < \epsilon}} f(z_i)(z_i - z_{i-1})$$

End of lecture 4. April 21, 2016

Some additional comments regarding the path integral are in order. We also allow curves with self-intersections. Let $\Omega \subset \mathbb{C}$ be open and $\gamma : [a, b] \to \Omega$ Lipschitz, i.e. there exists $L < \infty$ such that for all $t_1, t_2 \in [a, b]$ we have

$$|\gamma(t_2) - \gamma(t_1)| \leq L|t_2 - t_1|.$$

We want to allow curves with self-intersections; thus we are not asking γ to be injective.
Our Lipschitz assumption has several consequences. The function $\operatorname{Re}\gamma$ is of bounded variation:

$$\sup_{a<t_0<\cdots<t_N<b} |\operatorname{Re}\gamma(t_n) - \operatorname{Re}\gamma(t_{n-1})| < L|b-a|$$

and similarly for $\operatorname{Im}\gamma$. Both $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ are also absolutely continuous, i.e. for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{n=1}^N |\gamma(t_{2n}) - \gamma(t_{2n-1})| < \varepsilon \quad \text{if} \quad \sum_{n=1}^N |t_{2n} - t_{2n-1}| < \delta.$$

This implies differentiability almost everywhere with a derivative bounded in L^∞. We also have

$$\int_a^x (\operatorname{Re}\gamma(t))' dt = \operatorname{Re}\gamma(x) - \operatorname{Re}\gamma(a).$$

The same holds for $\operatorname{Im}\gamma$.

Definition 1.19. For $f : \Omega \to \mathbb{C}$ continuous and $\gamma : [a,b] \to \Omega$ Lipschitz we define

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt$$

Theorem 1.20. *Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ continuous and $\gamma : [a,b] \to \Omega$ Lipschitz. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all partitions $a = t_0 < \cdots < t_N = b$ with $|t_n - t_{n-1}| < \delta$ we have*

$$\left| \int_\gamma f(z)dz - \sum_{n=1}^N f(\gamma(t_n))(\gamma(t_n) - \gamma(t_{n-1}) \right| < \varepsilon.$$

Proof. We write

$$\begin{aligned}\int_\gamma f(z)dz &= \sum_{k=1}^N \int_{t_{n-1}}^{t_n} f(\gamma(t))\gamma'(t)dt \\ &= \sum_{n=1}^N \Big(\int_{t_{n-1}}^{t_n} f(\gamma(t_n))\gamma'(t)dt + \int_{t_{n-1}}^{t_n} (f(\gamma(t)) - f(\gamma(t_n)))\gamma'(t)dt \Big)\end{aligned}$$

The first term equals

$$\sum_{n=1}^N f(\gamma(t_n))(\gamma(t_n) - \gamma(t_{n-1}))$$

by the fundamental theorem of calculus for absolutely continuous functions. We can estimate the second term by exploiting uniform continuity of $f \circ \gamma$ on $[a, b]$. Namely, choose $\delta > 0$ small enough so that $|f(\gamma(t)) - f(\gamma(t'))| < \frac{\varepsilon}{L(b-a)}$ whenever $|t - t'| < \delta$. Here L is the Lipschitz constant of γ. Then we can estimate the error term as follows:

$$\left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (f(\gamma(t)) - f(\gamma(t_n)))\gamma'(t)dt \right| < \sum_{n=1}^{N} (t_n - t_{n-1}) \frac{\varepsilon}{b-a} = \varepsilon.$$

□

The path integral is invariant under reparametrization. Assume that $s : [a, b] \to [\widetilde{a}, \widetilde{b}]$ is monotonously increasing, bijective and the new path

$$\widetilde{\gamma} : [\widetilde{a}, \widetilde{b}] \to \mathbb{C},\ \gamma(t) = \widetilde{\gamma}(s(t)) \text{ for all } t \in [a, b]$$

is Lipschitz. Then $\int_\gamma f(z)dz = \int_{\widetilde{\gamma}} f(z)dz$. This can be shown by an appeal to the Riemann-Stieltjes sums from above (exercise).

Definition 1.21. Let $\Omega \subset \mathbb{C}$ be open. A function $f : \Omega \to \mathbb{C}$ is *holomorphic* in a point $z_0 \in \mathbb{C}$ if it is complex dfferentiable in a disc $D_R(z_0) \subset \Omega$.

The path integral leads to a simple way to exhibit (local) primitives of holomorphic functions. Let $\Omega = D_R(z_0)$ and f holomorphic, then there exists F with $F' = f$ and

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

because $F \circ \gamma$ is Lipschitz.

$$(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t).$$

The existence of a primitive depends on the topology of the domain (in fact it needs to be simply connected). For example, let $\Omega = \mathbb{C}\backslash\{0\}$. The function $f(z) = 1/z$ is holomorphic on Ω, but has no primitive on Ω.
We can exploit this property of holomorphic functions to define path integrals along curves $\gamma : [a, b] \to \Omega$ which are merely required to be continuous.

Definition 1.22. Let $f : \Omega \to \mathbb{C}$ be holomorphic and $\gamma : [a, b] \to \Omega$ continuous. We define the path integral $\int_\gamma f(z)dz$ as follows.

For all $t \in [a, b]$ we find δ_t and $\widetilde{\delta_t}$ such that $D_{\delta_t}(\gamma(t)) \subset \Omega$ and for all $\tilde{t}$ with $|\tilde{t} - t| < \widetilde{\delta_t}$ we have that $\gamma(\tilde{t}) \in D_{\delta_t}(\gamma(t))$. Since $[a, b]$ is compact we can select finitely many t_n such that the intervals $\left(t_n - \frac{\tilde{\delta}_{t_n}}{3}, t_n + \frac{\tilde{\delta}_{t_n}}{3}\right)$ cover $[a,b]$. Let $\delta = \min_n \frac{\delta_{t_n}}{3}$. For all $t \in [a,b]$ there is an n such that for all $|\tilde{t} - t| < \delta$ we have $\gamma(\tilde{t}) \in D_{\delta_{t_n}}(\gamma(t_n))$. Find a partition $a = s_0 < \cdots < s_N = b$ with $\max_n |s_n - s_{n-1}| < \delta$. Let F_n be a primitive of f on $D_{\delta_{t_n}}(\gamma(t_n))$. Now we can *define*

$$\int_\gamma f(z)dz := \sum_{n=1}^{N} F_n(\gamma(s_n)) - F_n(\gamma(s_{n-1})).$$

It remains to show that this definition is independent of the involved choices (exercise).
We turn our attention now to several very typical properties of holomorphic functions.

Theorem 1.23 (Mean value property)**.** *Let f holomorphic on $D_R(z_0)$. Then for $r < R$ we have*

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt = f(z_0).$$

Proof. Define

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

with the understanding that $g(z_0) = f'(z_0)$. Then g is also holomorphic on $D_R(z_0)$. Let $\gamma : [0, 2\pi] \to D_R(z_0)$, $\gamma(t) = z_0 + re^{it}$. Then, $\int_\gamma g(z)dz = 0$. That is,

$$\int_\gamma \frac{f(z)}{z - z_0}dz = \int_\gamma \frac{f(z_0)}{z - z_0}dz = f(z_0)\int_0^{2\pi} \frac{1}{re^{it}} ire^{it}dt = 2\pi i f(z_0).$$

On the other hand,

$$\int_\gamma \frac{f(z)}{z - z_0}dz = \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} ire^{it}dt = i\int_0^{2\pi} f(z_0 + re^{it})dt.$$

The claim follows. □

Theorem 1.24 (Maximum principle)**.** *Let $\Omega \subset \mathbb{C}$ be open and connected, f holomorphic on Ω. If $|f|$ assumes its maximum value at $z_0 \in \Omega$, then f is constant.*

In other words, non-constant holomorphic functions assume their maxima on the boundary of the domain of definition.

Proof. Let $f \not\equiv 0$. Define $g(z) = f(z)\frac{|f(z_0)|}{f(z_0)}$. Then $g(z_0) = |f(z_0)|$ and for all $z \in \Omega$,

$$\operatorname{Re} g(z) \leq g(z_0)$$

Consider $h(z) = g(z) - g(z_0)$. Then $\operatorname{Re} h(z) \leq 0$. Choose r with $\overline{D_r(z_0)} \subset \Omega$. By the mean value property,

$$0 = h(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} h(z_0 + re^{it})dt.$$

Since $\operatorname{Re} h$ is continuous and non-positive, we must have $\operatorname{Re} h(z_0 + re^{it}) = 0$ for all t. Also, $\operatorname{Re} h(z_0 + \tilde{r}e^{it}) = 0$ for all t, $\tilde{r} < r$. By the Cauchy-Riemann equations we obtain $\frac{\partial}{\partial x}\operatorname{Im} h = 0$ and $\frac{\partial}{\partial y}\operatorname{Im} h = 0$. Therefore h, and consequently also f, is constant in a neighborhood of z_0. Thus we proved that the non-empty set $\{z \in \Omega : f(z) = f(z_0)\}$ is open. By continuity of f, it is also closed so it must equal Ω because Ω is connected. □

Definition 1.25 (Entire functions)**.** A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is called *entire*.

Theorem 1.26 (Liouville)**.** *Let f be an entire function. If f is bounded, then it is constant.*

Proof. Consider $g(z) = \frac{f(z)-f(z_0)}{z-z_0}$, $g(z_0) = f'(z_0)$ for an arbitrary $z_0 \in \mathbb{C}$. Then g is again entire and for all $\varepsilon > 0$ such that for all $|z - z_0| > 1/\varepsilon$ we have

$$|g(z)| \leq C\varepsilon.$$

By the maximum principle, $|g(z)| \leq C\varepsilon$ for all $z \in \overline{D_{1/\varepsilon}(z_0)}$. Since ε was arbitrary, $g \equiv 0$. □

Theorem 1.27. *Let f be entire and bijective with holomorphic inverse. Then there exist $a, b \in \mathbb{C}$ such that*

$$f(z) = az + b.$$

Proof. Let z_0 be such that $f'(z_0) \neq 0$ (exists because f cannot be constant). Without loss of generality suppose that $z_0 = 0$ (by translating the function). Also assume that $f(0) = 0$ (by subtracting $f(0)$ from f). Then the function $h(z) = f(z)/z$, $h(0) = f'(0)$ is entire and vanishes nowhere (since $f(z) \neq 0$ for $z \neq 0$ by injectivity). Thus also

$$g(z) = \frac{1}{h(z)}$$

is an entire function. We claim that it is also bounded. By continuity of f^{-1}, there is $\varepsilon > 0$ such that for all $|\xi| < \varepsilon$, $|f^{-1}(\xi)| \leq 1$. Thus, for $|z| > 1$, $|f(z)| \geq \varepsilon$, so $|g(z)| \leq \frac{1}{\varepsilon}$. For $|z| \leq 1$ we have boundedness by continuity. By Liouville's theorem, g is a constant and the claim follows. □

Theorem 1.28. *Let $f : \Omega \to \mathbb{C}$ be holomorphic and non-constant. Assume $f(z_0) = 0$ for a given $z_0 \in \Omega$. Then there exists $\delta > 0$ with $f(z) \neq 0$ for all $z \in D_\delta(z_0) \backslash \{z_0\}$.*

This theorem shows that zeros of holomorphic functions are isolated.

Proof. Without loss of generality we assume $z_0 = 0$ (by translating the function). Write

$$f(z) = \sum_{n=N}^{\infty} a_n z^n = z^N \sum_{n=N}^{\infty} a_n z^{n-N}$$

with $a_N \neq 0$, $N \geq 1$. By continuity, there exists $\delta > 0$ such that $\sum_{n=N}^{\infty} a_n z^{n-N} \neq 0$ for all $z \in D_\delta(0)$. □

Theorem 1.29. *Let $f : \Omega \to \mathbb{C}$ be holomorphic and non-constant. Then f is* open *(i.e. $f(\Omega) \subset \mathbb{C}$ is an open set).*

Proof. Let $w_0 \in f(\Omega)$. Then there is $z_0 \in \Omega$ such that $f(z_0) = w_0$. We argue by contradiction and suppose that w_0 is not in the interior of $f(\Omega)$. Thus, for every $\varepsilon > 0$ there exists $\xi \in D_\varepsilon(w_0)$ such that $\xi \notin f(\Omega)$. By the previous theorem we pick δ such that $f(z) - w_0 \neq 0$ for $z \in D_\delta(z_0) \backslash \{z_0\}$. Let $0 < r < \delta$. The set $K = \{z_0 + re^{it} : t \in [0, 2\pi]\}$ is compact. Thus there exists $\varepsilon_0 > 0$ such that $|f(z) - w_0| > \varepsilon_0$ for all $z \in K$. Now take $\xi \in D_{\varepsilon_0/2}(w_0)$ such that $\xi \notin f(\Omega)$. Then the function

$$g(z) = \frac{1}{f(z) - \xi}$$

is holomorphic in Ω. For $z \in K$ we have

$$|g(z)| \leq \frac{1}{|f(z) - w_0| - |w_0 - \xi|} < \frac{1}{\varepsilon_0 - \varepsilon_0/2} = \frac{2}{\varepsilon_0}.$$

But,

$$|g(z_0)| = \frac{1}{|w_0 - \xi|} > \frac{2}{\varepsilon}.$$

This contradicts the maximum principle applied to g. □

End of lecture 5. April 25, 2016