Complex Analysis
Lecture notes

Prof. Dr. Christoph Thiele[*]
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Contents

1 Fundamentals

A complex number is a pair (x, y) of real numbers. The space $\mathbb{C} = \mathbb{R}^2$ of complex numbers is a two-dimensional $\mathbb{R}$-vector space. It is also a normed space with the norm defined as

$$|(x, y)| = \sqrt{x^2 + y^2}.$$

An additional feature that makes $\mathbb{C}$ very special is that it also has a product structure defined as follows.

Definition 1.1 (Product of complex numbers)**.** For two complex numbers $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$, their product is defined by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

This defines a map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$. It can be rewritten in terms of another product, the matrix product:

$$(x_1, y_1)(x_2, y_2) = (x_1, y_1)\begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix}.$$

[*]Notes by Joris Roos and Gennady Uraltsev.

In fact, we can embed the complex numbers into the space of real 2×2 matrices via the linear map

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{R}^{2\times 2} \\ (x, y) &\longmapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \end{aligned}$$

The map translates the product of complex numbers into the matrix product. This is very helpful to verify that the product of complex numbers is

1. commutative,
2. associative,
3. distributive, and
4. has a unit element:

   $$(x_1, y_1) = (1, 0)(x_1, y_1), \text{ and}$$

5. has inverses: if $(x, y) \neq 0$, then

   $$(x, y)\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) = \left(\frac{x^2 + y^2}{x^2 + y^2}, \frac{xy - yx}{x^2 + y^2}\right) = (1, 0).$$

   In terms of the matrix representation this property is just a restatement of the fact that

   $$\det \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x^2 + y^2 \neq 0 \tag{1.1}$$

   for $(x, y) \neq 0$.

Summarizing, the product of complex numbers gives $\mathbb{C}$ the structure of a field. The existence of such a product structure makes $\mathbb{R}^2$ unique among the higher dimensional Euclidean spaces $\mathbb{R}^d$, $d \geq 2$. Roughly speaking, the reason for this phenomenon is the very special structure of the above 2×2 matrices. In higher dimensions it becomes increasingly difficult to find a matrix representation such that Property 5 is satisfied. The only cases in which it is possible at all give rise to the quaternion ($d = 4$) and octonion ($d = 8$) product, neither of which is commutative (and the latter is not even associative).

Another important property is that we have compatability of the product with the norm:

$$|(x_1, y_1)(x_2, y_2)| = |(x_1, y_1)| \cdot |(x_2, y_2)|.$$

This is a consequence of the determinant product theorem and the identity

$$|(x, y)| = \sqrt{\det \begin{pmatrix} x & y \\ -y & x \end{pmatrix}}.$$

One consequence of this is that for fixed (x_1, y_1), the map $(x_1, y_1) \mapsto (x_1, y_1)(x_2, y_2)$ is continuous (but of course this can also be derived differently).
We now proceed to introduce the conventional notation for complex numbers.

Definition 1.2. We write $1 = (1, 0)$ to denote the multiplicative unit. $i = (0, 1)$ is called the *imaginary unit.* A complex number (x, y) is written as

$$z = x + iy.$$

$x =: \mathrm{Re}\,(z)$ is called the *real part* and $y =: \mathrm{Im}\,(z)$ the *imaginary part.* The *complex conjugate* of $z = x + iy$ is given by

$$\overline{z} = x - iy$$

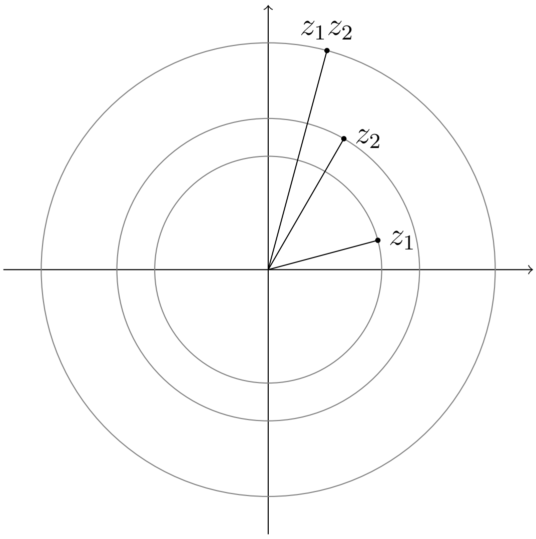
We have the following identities:

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1,$$

$$|z|^2 = z\overline{z} = (x + iy)(x - iy) = x^2 + y^2,$$

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

The product of complex numbers has a geometric meaning. Observe that the unit circle in the plane consists of those complex numbers z with $|z| = 1$. Say that z_1, z_2 lie on the unit circle. That is, $|z_1| = 1$, $|z_2| = 1$. Then also $|z_1 z_2| = |z_1| \cdot |z_2| = 1$, so also $z_1 z_2$ is on the unit circle. So the linear map $\mathbb{C} \to \mathbb{C}, z_1 \mapsto z_1 z_2$ maps the unit circle to itself. Recall that there are not too many linear maps with this property: only rotations and reflections. Since the determinant is positive by (1.1) it must be a rotation.

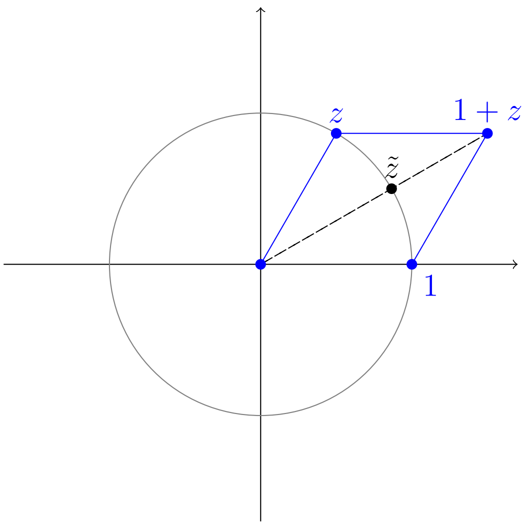


Every non-zero complex number can be written as the product of one on the circle and a real number:

$$z = \frac{z}{|z|}|z|$$

Multiplication with a real number corresponds to stretching, so we conclude from the above that multiplication with a complex number corresponds to a rotation and stretching of the plane.

Example 1.3. We use our recently gained geometric intuition to derive a curious formula for the square root of a complex number. Look at the following picture.



We have given some z with $|z| = 1$ and would like to find $\tilde{z}$ with $\tilde{z}^2 = z$. The picture suggests to pick

$$\tilde{z} = \frac{1+z}{|1+z|}.$$

Indeed we have

$$\tilde{z}^2 = \frac{(1+z)^2}{(1+z)(1+\overline{z})} = \frac{1+z}{1+\overline{z}} = \frac{z\overline{z}+z}{1+\overline{z}} = z\frac{1+\overline{z}}{1+\overline{z}} = z.$$

Now let $z \neq 0$ be a general complex number and apply the above to $\frac{z}{|z|}$. Then the square roots of z are given by

$$\sqrt{z} = \pm \frac{1+\frac{z}{|z|}}{\left|1+\frac{z}{|z|}\right|}\sqrt{|z|}.$$

We now turn our attention to functions of a complex variable $f : \mathbb{C} \to \mathbb{C}$. A prime example is given by complex power series:

$$\sum_{n=0}^{\infty} a_n z^n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n z^n.$$

To find out when this limit exists we check when the sequence of partial sums is Cauchy. Take $M < N$ and compute:

$$\left|\sum_{n=0}^{N} a_n z^n - \sum_{n=0}^{M} a_n z^n\right| = \left|\sum_{n=M+1}^{N} a_n z^n\right| \leq \sum_{n=M+1}^{N} |a_n z^n| = \sum_{n=M+1}^{N} |a_n|\, r^n,$$

where $r = |z|$. This implies that if $\sum_{n=0}^{\infty} |a_n| r^n$ converges in $\mathbb{R}$, then $\sum_{n=0}^{\infty} a_n z^n$ converges in $\mathbb{C}$. Next, $\sum_{n=0}^{\infty} |a_n| r^n < \infty$ holds if there exists $\tilde{r} > r$ with $\sup_n |a_n|\tilde{r}^n < \infty$ because

$$\sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} a_n \tilde{r}^n \left(\frac{r}{\tilde{r}}\right)^n \leq \left(\sup_n |a_n|\tilde{r}^n\right) \sum_{n=0}^{\infty} \left(\frac{r}{\tilde{r}}\right)^n < \infty.$$

Definition 1.4. The *convergence radius* of a power series $\sum_{n=0}^{\infty} a_n z^n$ is defined as

$$R := \sup\{\tilde{r} \,:\, \sup_n |a_n|\tilde{r}^n < \infty\}.$$

- For $z \in D_R(0) = \{z \,:\, |z| < R\}$, the sum $\sum_{n=0}^{\infty} a_n z^n$ converges.
- For $|z| > R$, the sum $\sum_{n=0}^{\infty} a_n z^n$ diverges.

- For $|z| = R$ both convergence and divergence are possible.

Examples 1.5. The *exponential series*

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

has convergence radius $R = \infty$. The same holds for

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n},$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

These combine to give the Euler formula,

$$e^{iz} = \cos(z) + i\sin(z).$$

For φ real, $e^{i\varphi}$ lies on the unit circle. Let us see how to derive this from the definition. For n large we have

$$e^{i\varphi} = \left(e^{i\frac{\varphi}{n}}\right)^n \sim \left(1 + i\frac{\varphi}{n}\right)^n,$$

$$\left|1 + i\frac{\varphi}{n}\right| \leq 1 + \left(\frac{\varphi}{n}\right)^2.$$

Notice that

$$\left(1 + \frac{\varphi^2}{n^2}\right)^{n^2} \overset{n\to\infty}{\longrightarrow} e^{\varphi^2},$$

so

$$|e^{i\varphi}| \sim \left|1 + i\frac{\varphi}{n}\right|^n \sim \sqrt[n]{e^{\varphi^2}} \to 1.$$

Remark 1.6. General polynomials in x, y on $\mathbb{R}^2$ are of the form

$$\sum_{n,m=0}^{N} a_{n,m} x^n y^m = \sum_{n,m=0}^{N} a_{n,m} \left(\frac{z+\overline{z}}{2}\right)^n \left(\frac{z-\overline{z}}{2i}\right)^m = \sum_{n,m=0}^{N} b_{n,m} z^n \overline{z}^m.$$

In complex analysis we only consider the case $b_{n,m} = 0$ for $m \neq 0$.

Definition 1.7. A function $f : \mathbb{C} \to \mathbb{C}$ is called *complex differentiable* at $z \in \mathbb{C}$if for $h \in \mathbb{C}$ with $|h|$ small enough we have

$$f(z+h) = f(z) + hg(z) + o(h) \tag{1.2}$$

such that for all $\varepsilon > 0$ there exists $\delta > 0$ with $|o(h)| < \varepsilon|h|$ for all $|h| < \delta$.

Theorem 1.8. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has convergence radius R, then*

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

also has convergence radius R and for $|z| < R$, f is complex differentiable at z.

Proof. We already know how to differentiate power series from real analysis. The proof of this theorem works exactly the same way as in the real case:

$$\begin{aligned} f(z+h) &= \sum_{n=0}^{\infty} a_n (z+h)^n = \sum_{n=0}^{\infty} \left(a_n z^n + nhz^{n-1} + \sum_{k=2}^{n} a_n \binom{n}{k} h^k z^{n-k} \right) \\ &= f(z) + hg(z) + o(h) \end{aligned}$$

and

$$\left| \frac{o(h)}{h} \right| \leq |h| \sum_{n=0}^{\infty} |a_n| n^2 \sum_{k=0}^{n-2} \binom{n+2}{k} |h|^k |z|^{n+2-k} \leq |h| \sum_{n=0}^{\infty} |a_n| n^2 (|z| + |h|)^{n+2}.$$

□

Compare this to the real Taylor series in $\mathbb{R}^2$: let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be totally differentiable in z, then there exists a matrix A with

$$f(z+h) = f(z) + Ah + o(h) \tag{1.3}$$

and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|o(h)| \leq \varepsilon |h|$ for $|h| < \delta$. Note that the product in (1.3) is the matrix product and the product in (1.2) is the product of complex numbers. They coincide if and only if

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Thus we find that a function $f(z) = (u(x,y), v(x,y))$ that is (real) totally differentiable at z is complex differentiable at z if and only if

$$\frac{\partial u}{\partial x}(z) = \frac{\partial v}{\partial y}(z) \quad \text{and} \quad \frac{\partial u}{\partial y}(z) = -\frac{\partial v}{\partial x}(z). \tag{1.4}$$

These are called the *Cauchy-Riemann differential equations.*

End of lecture 1. April 11, 2016

We will now study some properties of functions on a complex disk. In particular we will concentrate on the question of regularity and differentiability. In the previous lecture we have mentioned that power series are complex differentiable inside the disk of the radius of convergence. To establish notation let us introduce the following sets.

A We denote by A the set of all power series

$$A := \left\{ \sum_{n=0}^{\infty} a_n z^n \; : \; a_n \in \mathbb{C}, z \in D_R(0) \right\}$$

with radius of convergence at least $R > 0$ so that for all z in the domain $D_R(0) = \{z \, : \, |z - 0| < R\}$ the series converges absolutely (equivalently $sup_n |a_n| \tau^n < \infty$ for any $0 \leq \tau < R$).

As noted previously, A is a subset of the set of all formal power series on $D_R \subset \mathbb{C}$ given by $\sum_{m,n=0}^{\infty} b_{n,m} x^n y^m$ with $x = \mathrm{Re}\,(z)$, $y = \mathrm{Im}\,(z)$. Equivalently these formal series can be expressed as $\sum_{n,m=0}^{\infty} a_{n,m} z^n \bar{z}^m$ and A puts both a restriction of the growth of the coefficients $a_{n,m}$ given by the condition of being convergent on $D_R(0)$ and the additional constraint that $a_{n,m} = 0$ unless $m = 0$.

B We denote by B the set of functions that are complex differentiable in every point of the open disk $D_R(0)$. In particular, as per condition (1.2), B consists of those functions $f : D_R(0) \mapsto \mathbb{C}$ such that for any point $z \in D_R(0)$ and for any increment $h \, : \, |z| + |h| < R$ there exists the complex derivative $g(z) \in \mathbb{C}$ i.e. a complex coefficient such that

$$f(z + h) = f(z) + hg(z) + o(h)$$

where $o(h)$ is some function (depending on z) for which for any $\epsilon > 0$ there exists a $\exists \delta > 0$ such that for any $|h| < \delta$, $|z| + |h| < R$ we have that $o|h| \leq \epsilon |h|$. Recall that this is related to total differentiability on $\mathbb{C} \equiv \mathbb{R}^2$. As a matter one can write the following for a totally differentiable function on $\mathbb{C}$:

$$f(z + h) = f(z) + A(z)h + o(h) \qquad A(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}.$$

Complex differentiability is equivalent to asking the differential as a linear map $A : \mathbb{R}^2 \mapsto \mathbb{R}^2$ can be represented by complex multiplication: $A(z)h = g(z)h$ for some $g(z) \in \mathbb{C}$. This holds if and only if $a(z) = d(z)$ and $b(z) = -c(z)$.

Let us recall the Cauchy-Riemann equations (1.4) and elaborate how they are related to complex differentiability. Setting $f(z) = (u(x, y),\, v(x, y))$, the equations are given by

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \qquad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}.$$

We can rewrite this equation by defining the following two crucial differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ by setting

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Once again setting $f(x,y) = u(x,y) + iv(x,y)$ we can compute

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) = 0$$

It is apparent that the two Cauchy-Riemann equations are just the real and imaginary part of $\frac{\partial f}{\partial \bar{z}}$. Since we have already mentioned that complex differentiability is equivalent to a condition on the differential matrix A that corresponds to the Cauchy-Riemann equations in terms of partial derivatives, it follows that a function is complex differentiable if and only if it is totally differentiable and has $\frac{\partial f}{\partial \bar{z}} = 0$. Furthermore, if f is complex differentiable then we write

$$f'(z) := \frac{\partial}{\partial z} f(z).$$

Finally, in terms of the the real and imaginary part separately we have

$$\frac{\partial}{\partial z} f(z) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right).$$

C We denote by C the subset of continuous functions $f : D_R(0) \mapsto \mathbb{C}$ such that the following integral condition holds

$$\int_{(a,b,c)} f(z)dz = 0 \qquad \forall a,b,c \in D_R(0).$$

Here (a,b,c) is the (oriented) boundary of the (oriented) triangle, also referred to as a simplex, formed by the points a, b, and c. We will identify (a,b,c) by the closed path composed of the three segments $a \to b \to c \to a$. We recall the definition of the integral along a path of a complex function. For now we restrict ourselves to the case were the support of the path is a complex segment.

Definition 1.9 (Integral of a complex function along a segment)**.** Consider the segment (a, b) with $a, b \in \mathbb{C}$ and a complex-valued continuous function $f : \Omega \subset \mathbb{C} \mapsto \mathbb{C}$ defined on an open neighborhood of (a, b). We set the integral of the function f along (a, b) to be

$$\int_{(a,b)} f(z)dz := \int_0^1 f\left(bt + a(1-t)\right)(b-a)dt.$$

Here the integrand on the right hand side is a function $[0, 1] \mapsto \mathbb{C} \equiv \mathbb{R}^2$ and the integral is simply calculated coordinate-wise. Notice however that the integrand itself $f\left(bt + a(1-t)\right) \cdot (b-a)$ is expressed itself as a *complex* product.

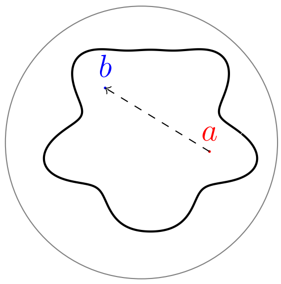


Figure 1: A segment defining a path from a to b.

This definition of the integral over a segment corresponds to the well known concept of a path integral, and extends it to complex functions:

$$\int_{(a,b)} f(z)dz = \int_0^1 f\left(bt + a(1-t)\right)(b-a)dt = \int_\gamma f d\gamma = \int_0^1 f(\gamma(t))\gamma'(t)dt$$

with $\gamma(t) = bt + a(1-t)$ as the path that parameterizes the segment. We naturally extend this definition to the three oriented segments of the boundary of a triangle by setting

$$\int_{(a,b,c)} f(z)dz := \int_{(a,b)} f(z)dz + \int_{(b,c)} f(z)dz + \int_{(c,a)} f(z)dz.$$

Finally notice that the definition of integrating along a path is oriented and as such we have

$$\int_{(a,b)} f(z)dz = -\int_{(b,a)} f(z)dz.$$

This can be easily verified by a change of variables.

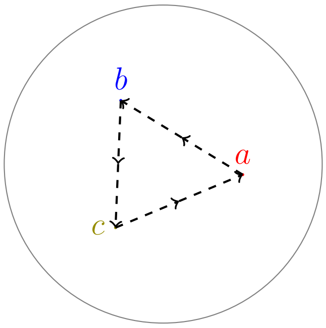


Figure 2: A triangle and its oriented boundary.

The characterization of the set C in terms of path integrals is geometric and does not rely on the smoothness of f. As a matter of fact we require f merely to be continuous. However we will now see that integral over triangles condition over all possible triangles implies stronger structure results and in particular that f is actually smooth and complex differentiable.

Theorem 1.10. *The classes of functions we introduced coincide i.e.* $A = B = C$*.*

$\mathbf{A} \subset \mathbf{B}$ The rules of differentiation of power series imply immediately the inclusion $A \subset B$.
$\mathbf{A} \subset \mathbf{C}$ We will now show directly that the path integral of a power series along a closed path, and specifically (a, b, c) is zero. In previous courses of analysis we have seen a similar statement for gradient fields and the proof followed from the existence of a primitive. We can, however, deduce the existance of a primitive of a power series formally and this will provide us with the needed elements to adapt a similar approach.
Recall the definition of the set A: $f \in A$ is of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let us define its primitive via

$$F(z) := \sum_{n=0}^{\infty} \frac{1}{n+1} a_n z^{n+1}.$$

Clearly $F \in A$ since it is a power series and its radius of convergence is not smaller than that of f. This follows simply from the bound on the n^{th} coefficient of F by that of f:

$$\frac{1}{n+1}|a_n| \leq |a_n|.$$

We claim that F is effectively a primitive of f and in particular

$$\int_{(a,b)} f(z)dz = \int_0^1 f\left(a(1-t) + bt\right)(b-a)dt = F(b) - F(a).$$

The first equality is just the definition of a complex path integral. To show the second equality let us define $g(y) = F(\gamma(t))$ with $\gamma(t) = a(1-t)+bt$ and let us show that

$$g'(t) = f\left(a(1-t)+bt\right)(b-a).$$

This is essentially the chain rule for complex-valued complex differentiable functions. We write

$$\begin{aligned} g(t+h) &= F(\gamma(t+h)) = F\left(\gamma(t)+(b-a)h\right) \\ &= F(\gamma(t)) + (b-a)hf\left(\gamma(t)\right) + o\left((b-a)h\right) \\ &= g(t) + (b-a)hf\left(\gamma(t)\right) + o\left((b-a)h\right) \end{aligned}$$

Here we used that the complex differential of F in $\gamma(t)$ is given by $f\left(\gamma(t)\right)$ and that $(b-a)h$ is a small complex increment. Notice also that h is a real increment. We have thus that

$$\int_0^1 f\left(a(1-t)+bt\right)(b-a)dt = F(b) - F(a)$$

and

$$\begin{aligned} \int_{(a,b,c)} f(z)dz = & \int_{(a,b)} f(z)dz + \int_{(b,c)} f(z)dz + \int_{(c,a)} f(z)dz \\ = & F(b) - F(a) + F(c) - F(b) + F(a) - F(c) = 0 \end{aligned}$$

B $\subset$ **C** This statement is known as “Theorem from Goursat”. Let $f \in B$ be complex differentiable in $D_R(0)$. We must show that for any $\tilde{r} < R$ and $\forall a, b, c \in \overline{D_{\tilde{r}}(0)}$ one has $\int_{(a,b,c)} f(z)dz = 0$. It is sufficient to show that for any $\epsilon > 0$ and $\forall a, b, c \in D_{\tilde{r}}(0)$ we have that

$$\left| \int_{(a,b,c)} f(z)dz \right| \leq \epsilon \max\left(|b-a|, |c-b|, |a-c|\right)^2.$$

The argument we present relies on an induction on scales. The term

$$\max\left(|b-a|, |c-b|, |a-c|\right)^2$$

on the right hand side of the above entry is a measure of the scale of “how large” or the *scale* of the triangle. We will show that the statement holds for triangles that have sufficiently small scale and then to an induction argument that will show that is the statement holds for a certain scale it also holds for triangle up to twice as large. This would allow us to conclude the statement for all triangles.

Part 1: We start by showing that the above bound holds for all points $a, b, c \in D_{\tilde{r}}(0)$ with $\max(|b-a|, |c-b|, |a-c|) < \delta_{min}$ for some $\delta_{min} > 0$. For any $z \in D_{\tilde{r}}(0)$ there exists δ such that $\forall |h| < \delta$ we have

$$f(z+h) = f(z) + hf'(z) + o(h) \qquad \text{with } |o(h)| < \frac{\epsilon}{8}|h|.$$

Reasoning by compactness we can find a finite set $z_1, \ldots, z_N$ such that $\overline{D_{\tilde{r}}(0)} \subset \bigcup_{j-1}^{N} D_{\delta(z_i)/3}(z_i)$ where $\delta(z_i)$ is the radius for which the above bound holds. Setting $\delta_{min} := \frac{\min_i \delta(z_i)}{3}$ one has that $\forall z \in \overline{D_{\tilde{r}}(0)}$ $\forall |h| < \delta_{min}$ we have via the triangle inequality

$$f(z+h) = f(z) + hf'(z) + o(h) \qquad \text{with } |o(h)| < \frac{\epsilon}{4}|h|.$$

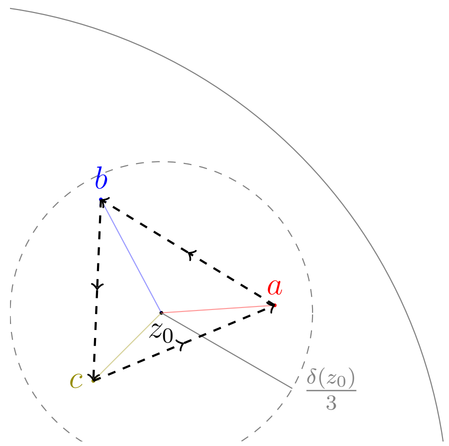


Figure 3: A triangle in a small circle

Now consider two point a, b with $|b-a| < \delta_{min}$. We can evaluate the contribution of the three terms of the expansion of f to the line integral.

$$\int_{(a,b)} f(z)dz = \int_{(a,b)} f(z_0) + (z-z_0)f'(z_0) + o(z-z_0)dz$$

The first term gives

$$\int_0^1 f(z_0)(b-a)dt = f(z_0)(b-a).$$

The second term gives

$$\begin{aligned}
&\int_0^1 f'(z_0)\left(bt - a(1-t) - z_0\right)(b-a)dt \\
=&f'(z_0)\left(\frac{1}{2}(b+a)(b-a) - a(b-a) - z_0(b-a)\right) \\
=&f'(z_0)\left(\frac{1}{2}(b^2-a^2) + z_0(b-a)\right)
\end{aligned}$$

We have crucially used complex differentiability of f here. As a matter of fact the algebraic manipulation relied on the commutativity of complex multiplication. If f were just any totally differentiable function then $f'(z_0)$ would be substituted by some arbitrary 2×2 matrix and the above identity would not hold.
Summing up the contributions of the three terms we obtain

$$\begin{aligned}
&\int_{(a,b,c)} f(z)dz = \int_{(a,b)} f(z)dz + \int_{(b,c)} f(z)dz + \int_{(c,a)} f(z)dz = \\
&= f(z_0)(b-a+c-b+a-c) \\
&+ f'(z_0)\left(\frac{1}{2}(b^2-a^2+c^2-b^2+a^2-c^2) + z_0(b-a+c-b+a-c)\right) \\
&+ \int_{(a,b,c)} o(z-z_0)dz
\end{aligned}$$

All terms except the last are null while for the last we have the bound

$$\begin{aligned}
&\left|\int_{(a,b,c)} f(z)dz\right| = \left|\int_{(a,b,c)} o(z-z_0)dz\right| \\
&< \frac{\epsilon}{4}\left(|b-a| + |c-b| + |a-c|\right) \max_{z\in(a,b,c)} |z-z_0| \\
&< \frac{3\epsilon}{4} \max\left(|b-a|, |c-b|, |a-c|\right)^2
\end{aligned}$$

as required.
Part 2: We have now proved that the bound we seek holds for triangles that are small enough. In particular we require that $\max(|b-a|, |c-b|, |a-c|) < \delta_{min}$. We will now show an inductive procedure that shows that if the statement holds for when $\max(|b-a|, |c-b|, |a-c|) < \delta$ then the same is true if $\max(|b-a|, |c-b|, |a-c|) < 2\delta$.
The main idea is given by decomposing a triangle into smaller triangles in a uniform way.

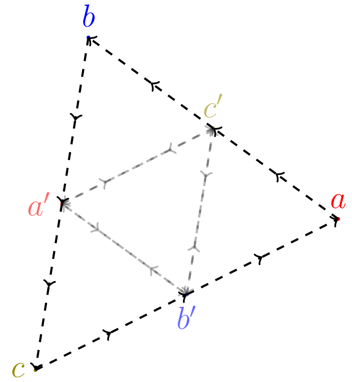


Figure 4: Decomposing triangles into smaller ones

To do so we use the median points as shown in figure 4. Let a', b', c' be the median points of the sides of (a, b, c) opposite of the respective vertices. We have

$$\int_{(a,b,c)} f(z)dz = \int_{(a,c',b')} f(z)dz + \int_{(b,a',c')} f(z)dz + \int_{(c,b',a')} f(z)dz + \int_{(a',b',c')} f(z)dz$$

$$\left|\int_{(a,b,c)} f(z)dz\right| < \epsilon \Big(\max\left(|c'-a|, |b'-c'|, |a-b'|\right)^2 + \max\left(...\right)^2 + \max\left(...\right)^2 + \max\left(|b'-a'|, |c'-b'|, |a'-c'|\right)^2\Big) < 4\epsilon \frac{\max\left(|b-a|, |c-b|, |a-c|\right)^2}{4}$$

as required. The crucial observation is that once we divide by the medians we obtain four triangles for which the largest of side lengths is bounded by a small $(1/2)$ factor of the lengths of the original triangle. This implies that first of all we may apply the assumptions at previous scale and that we obtain a bound with the same constant.

End of lecture 2. April 14, 2016

We will prove the following stronger version of Goursat's theorem.

Theorem 1.11. *Let $z_0 \in D_R(0)$, $f : D_R(0) \to \mathbb{C}$ continuous and complex differentiable in all points of $D_R(0)\backslash\{z_0\}$. Then $f \in C$.*

Proof. It suffices to show that for all $\tilde{r} < R$, $a, b, c \in D_{\tilde{r}}(0)$ we have

$$\int_{(a,b,c)} f(z)dz = 0.$$

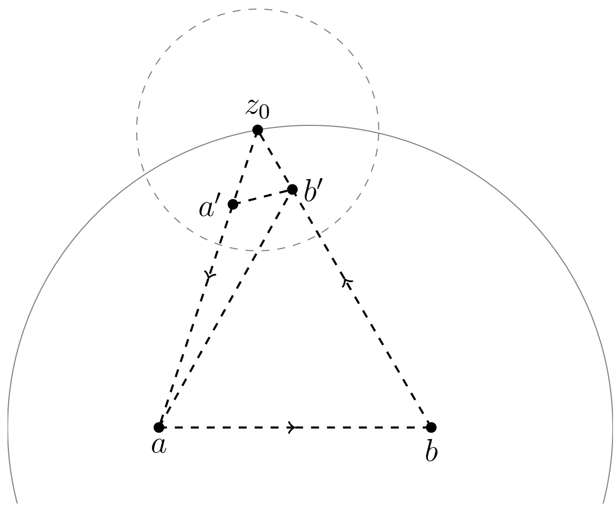
Let $10\delta = R - \tilde{r}$. By the same argument as in the proof of Goursat's theorem it suffices to show this for small triangles: for all $a, b, c \in \overline{D_{\tilde{r}}(0)}$ with $\max(|a - b|, |b - c|, |c - a|) \leq \delta/10$.
Case 1. $z_0 \notin D_{\delta/3}(a)$. Then $\int_{(a,b,c)} f(z)dz = 0$ holds by Goursat's theorem.
Case 2. $z_0 \in D_{\delta/3}(a)$. It suffices to show $\int_{(a,b,z_0)} f(z)dz = 0$ because

$$\int_{(a,b,c)} = \int_{(a,b,z_0)} + \int_{(b,c,z_0)} + \int_{(c,a,z_0)}.$$

We can also assume that the angle at z_0 is acute (if it is not acute, we bisect the angle at z_0 and consider the two resulting triangles). Next, construct a circle through z_0 that contains (a, b, z_0). We can do this such that the radius is at most δ.
Let $\varepsilon > 0$ be arbitrary. We will show

$$\left| \int_{(a,b,z_0)} f(z)dz \right| \leq \varepsilon.$$

By continuity of f at z_0 we can choose points a' on (a, z_0) and b' on (b, z_0) such that $|f(z) - f(z_0)| < \varepsilon/(3\delta)$ for all z on the triangle (a', b', z_0).



By Goursat's theorem we have

$$\int_{(a,b,b')} f(z)dz = \int_{(a',a,b')} f(z)dz = 0$$

so that

$$\int_{(a,b,z_0)} f(z)dz = \int_{(a',b',z_0)} f(z)dz.$$

We estimate,

$$\left|\int_{(a',b',z_0)} f(z)dz\right| = \left|\int_{(a',b',z_0)} f(z) - f(z_0)dz\right| \leq \underbrace{\int_0^1 |f(b't + a'(1-t)) - f(z_0)||b' - a'|dt + \cdots}_{<\varepsilon}$$

□

As a precursor to showing $B \subset A$ we first prove the following.

Theorem 1.12 (Cauchy)**.** *Let $f : D_R(0) \to \mathbb{C}$ complex differentiable on $D_R(0)$. Then for all $z_1 \in D_R(0)$ there exists $\delta > 0$ such that f can be represented by a convergent power series on $D_\delta(z_0) \subset D_R(0)$.*

Remark 1.13. In particular, this entails that functions which are complex differentiable in a neighborhood are automatically infinitely often complex differentiable.

Proof. For $w \in D_R(0)$ we consider the function

$$g_w(z) = \frac{f(z) - f(w)}{z - w}$$

with the understanding that $g_w(w) = f'(w)$. This function is continuous on $D_R(0)$ and complex differentiable on $D_R(0)\backslash\{w\}$. Continuity of g_w in w is a consequence of complex differentiability of f in w. Complex differentiability of g_w in $D_R(0)\backslash\{w\}$ follows by the product rule since $f(z) - f(w)$ and $\frac{1}{z-w}$ are both complex differentiable. Let us show the complex differentiability of $\frac{1}{z}$ on $\mathbb{C}\backslash\{0\}$ directly from the definition:

$$\frac{1}{z+h} - \frac{1}{z} = \frac{z-(z+h)}{z(z+h)} = \frac{-h}{z^2} + \frac{h}{z^2} - \frac{h}{z(z+h)} = -\frac{h}{z^2} + \frac{h^2}{z^2(z+h)} = \frac{-h}{z^2} + o(h)$$

where $o(h) = h^2/(z^2(z+h))$ so that

$$|o(h)| \leq |h^2| \left|\frac{1}{z^2(z+h)}\right| \leq |h|^2 \left|\frac{2}{z^3}\right|.$$

provided that $|h| < \frac{|z|}{2}$.

Choose $a, b, c \in D_R(0)$ such that z_0 lies in the interior of the triangle (a, b, c). Further, pick $\delta > 0$ small enough so that the circle of radius 2δ around z_0 is contained in the interior of the triangle (a, b, c).
Theorem 1.11 yields

$$\int_{(a,b,c)} g_w(z)dz = 0$$

for all $w \in D_\delta(z_0)$. That is,

$$\int_{(a,b,c)} \frac{f(z)}{z-w}dz = \left(\int_{(a,b,c)} \frac{dz}{z-w}\right) f(w)$$

Our claim is that

$$\int_{(a,b,c)} \frac{dz}{z-w} = \pm 2\pi i, \tag{1.5}$$

where the sign is according to whether the triangle (a, b, c) is oriented counter-clockwise $(+)$ or clockwise $(-)$. For the remainder of this proof, let us assume it is oriented counter-clockwise. We defer the proof of this claim to the end and first show how to use the equality

$$f(w) = \frac{1}{2\pi i} \int_{(a,b,c)} \frac{f(z)}{z-w}dz$$

to develop f into a convergent power series. The crucial point here is that on the right hand side, the free variable w no longer occurs inside the argument of f. Therefore we just need to know how to develop $w \mapsto \frac{1}{z-w}$ into a power series around z_0:

$$\frac{1}{z-w} = \frac{1}{(z-z_0)(w-z_0)} = \frac{1}{z-z_0} \cdot \frac{1}{1-\frac{w-z_0}{z-z_0}} = \frac{1}{z-z_0} \sum_{n=0}^{\infty} \left(\frac{w-z_0}{z-z_0}\right)^n.$$

As a consequence,

$$\begin{aligned}\int_{(a,b,c)} \frac{f(z)}{z-w}dz &= \int_{(a,b,c)} \frac{1}{z-z_0} f(z) \sum_{n=0}^{\infty} \left(\frac{w-z_0}{z-z_0}\right)^n dz \\ &= \sum_{n=0}^{\infty} \left(\int_{(a,b,c)} \frac{f(z)}{(z-z_0)^{n+1}}dz\right)(w-z_0)^n dz,\end{aligned}$$

where the interchange of integration and summation is justified by uniform convergence of the power series since $2|w-z_0| < 2\delta < |z-z_0|$ by construction.
It remains to prove (1.5). For starters we calculate

$$\int_{(a,b)} \frac{1}{z-w}dz = \int_0^1 \frac{b-a}{(b-a)t+a-w}dt = \int_0^1 \frac{1}{t+\frac{a-w}{b-a}}dt.$$

Temporarily denote $\frac{a-w}{b-a} = x + iy$ with x, y real numbers. Decompose the integral into real and imaginary part:

$$\int_0^1 \frac{1}{t+x+iy}dt = \int_0^1 \frac{(t+x)-iy}{(t+x)^2+y^2}dt = \int_0^1 \frac{t+x}{(t+x)^2+y^2}dt + i\int_0^1 \frac{-y}{(t+x)^2+y^2}dt.$$

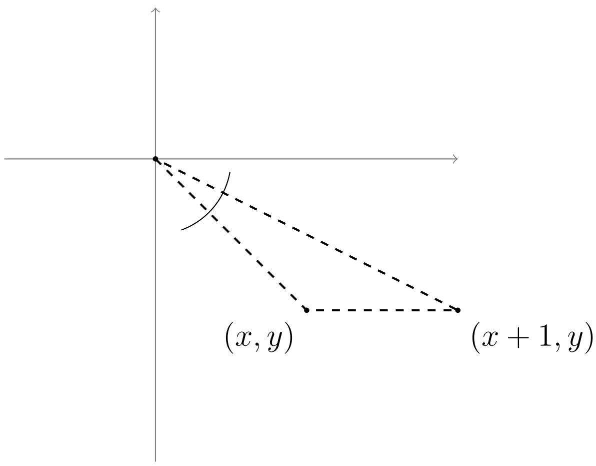
Now we are only dealing with two real integrals that we can evaluate. The first equals

$$\frac{1}{2}\int_x^{x+1} \frac{2t}{t^2+y^2}dt = \frac{1}{2}\left(\log((x+1)^2+y^2) - \log(x^2+y^2)\right) = \log\frac{\sqrt{(x+1)^2+y^2}}{\sqrt{x^2+y^2}}.$$

The second equals

$$-\int_x^{x+1} \frac{y}{t^2+y^2}dt = -\int_{x/y}^{(x+1)/y} \frac{1}{s^2+1}ds = -\arctan\left(\frac{x+1}{y}\right) + \arctan\left(\frac{x}{y}\right). \tag{1.6}$$

$\square$



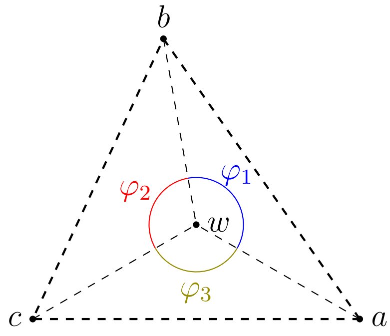
The angle at 0 in the triangle $(0, x+iy, x+1+iy)$ equals $\pm$(1.6). Since addition and multiplication with complex numbers preserves angles, that angle equals the angle at w in the triangle (w, a, b) (the two triangles are similar).
For the same reason we have

$$\log\frac{|(x+1,y)|}{|(x,y)|} = \log\frac{|b-w|}{|a-w|}.$$

Applying the same reasoning to the other two segments (b, c), (c, a) we get

$$\int_{(a,b,c)} \frac{1}{z-w}dz = \overbrace{\log\left(\frac{|b-w|}{|a-w|}\frac{|c-w|}{|b-w|}\frac{|a-w|}{|c-w|}\right)}^{=0} + i(\varphi_1+\varphi_2+\varphi_3) = 2\pi i.$$

The last equality is by inspection of the figure:



End of lecture 3. April 18, 2016