Complex Analysis
Lecture notes

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Summer term 2016
Universität Bonn

April 16, 2016

Contents

1 Fundamentals

A complex number is a pair (x, y) of real numbers. The space $\mathbb{C} = \mathbb{R}^2$ of complex numbers is a two-dimensional $\mathbb{R}$-vector space. It is also a normed space with the norm defined as

$$|(x, y)| = \sqrt{x^2 + y^2}.$$

An additional feature that makes $\mathbb{C}$ very special is that it also has a product structure defined as follows.

Definition 1.1 (Product of complex numbers)**.** For two complex numbers $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$, their product is defined by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

This defines a map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$. It can be rewritten in terms of another product, the matrix product:

$$(x_1, y_1)(x_2, y_2) = (x_1, y_1) \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix}.$$

[*]Notes by Joris Roos and Gennady Uraltsev.

In fact, we can embed the complex numbers into the space of real 2×2 matrices via the linear map

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{R}^{2\times 2} \\ (x,y) &\longmapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \end{aligned}$$

The map translates the product of complex numbers into the matrix product. This is very helpful to verify that the product of complex numbers is

1. commutative,
2. associative,
3. distributive, and
4. has a unit element:

$$(x_1, y_1) = (1,0)(x_1, y_1), \text{ and}$$

5. has inverses: if $(x, y) \neq 0$, then

$$(x,y)\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = \left(\frac{x^2+y^2}{x^2+y^2}, \frac{xy-yx}{x^2+y^2}\right) = (1,0).$$

In terms of the matrix representation this property is just a restatement of the fact that

$$\det \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x^2 + y^2 \neq 0 \tag{1.1}$$

for $(x, y) \neq 0$.

Summarizing, the product of complex numbers gives $\mathbb{C}$ the structure of a field. The existence of such a product structure makes $\mathbb{R}^2$ unique among the higher dimensional Euclidean spaces $\mathbb{R}^d$, $d \geq 2$. Roughly speaking, the reason for this phenomenon is the very special structure of the above 2×2 matrices. In higher dimensions it becomes increasingly difficult to find a matrix representation such that Property 5 is satisfied. The only cases in which it is possible at all give rise to the quaternion ($d = 4$) and octonion ($d = 8$) product, neither of which is commutative (and the latter is not even associative).

Another important property is that we have compatability of the product with the norm:

$$|(x_1, y_1)(x_2, y_2)| = |(x_1, y_1)| \cdot |(x_2, y_2)|.$$

This is a consequence of the determinant product theorem and the identity

$$|(x, y)| = \sqrt{\det \begin{pmatrix} x & y \\ -y & x \end{pmatrix}}.$$

One consequence of this is that for fixed (x_1, y_1), the map $(x_1, y_1) \mapsto (x_1, y_1)(x_2, y_2)$ is continuous (but of course this can also be derived differently).
We now proceed to introduce the convential notation for complex numbers.

Definition 1.2. We write $1 = (1, 0)$ to denote the multiplicative unit. $i = (0, 1)$ is called the *imaginary unit*. A complex number (x, y) is written as

$$z = x + iy.$$

$x =: \operatorname{Re}(z)$ is called the *real part* and $y =: \operatorname{Im}(z)$ the *imaginary part*. The *complex conjugate* of $z = x + iy$ is given by

$$\overline{z} = x - iy$$

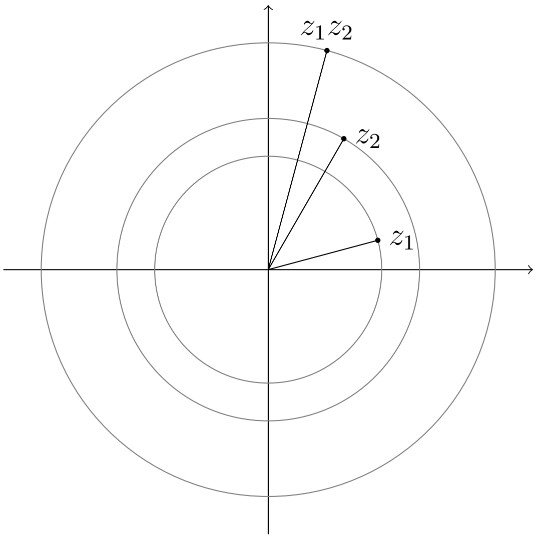
We have the following identities:

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1,$$

$$|z|^2 = z\overline{z} = (x + iy)(x - iy) = x^2 + y^2,$$

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

The product of complex numbers has a geometric meaning. Observe that the unit circle in the plane consists of those complex numbers z with $|z| = 1$. Say that z_1, z_2 lie on the unit circle. That is, $|z_1| = 1$, $|z_2| = 1$. Then also $|z_1 z_2| = |z_1| \cdot |z_2| = 1$, so also $z_1 z_2$ is on the unit circle. So the linear map $\mathbb{C} \to \mathbb{C}, z_1 \mapsto z_1 z_2$ maps the unit circle to itself. Recall that there are not too many linear maps with this property: only rotations and reflections. Since the determinant is positive by (1.1) it must be a rotation.

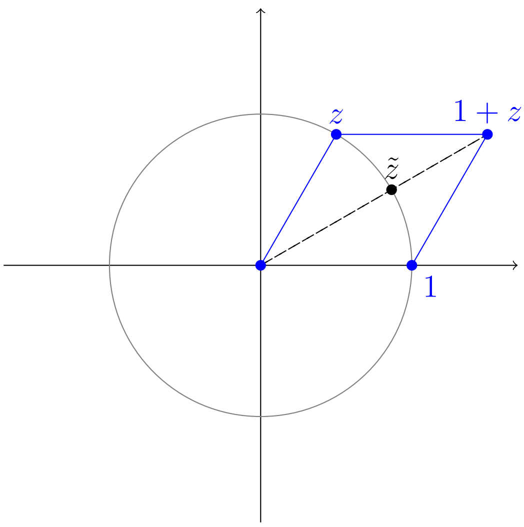


Every non-zero complex number can be written as the product of one on the circle and a real number:

$$z = \frac{z}{|z|}|z|$$

Multiplication with a real number corresponds to stretching, so we conclude from the above that multiplication with a complex number corresponds to a rotation and stretching of the plane.

Example 1.3. We use our recently gained geometric intuition to derive a curious formula for the square root of a complex number. Look at the following picture.



We have given some z with $|z| = 1$ and would like to find $\tilde{z}$ with $\tilde{z}^2 = z$. The picture suggests to pick

$$\tilde{z} = \frac{1+z}{|1+z|}.$$

Indeed we have

$$\tilde{z}^2 = \frac{(1+z)^2}{(1+z)(1+\overline{z})} = \frac{1+z}{1+\overline{z}} = \frac{z\overline{z}+z}{1+\overline{z}} = z\frac{1+\overline{z}}{1+\overline{z}} = z.$$

Now let $z \neq 0$ be a general complex number and apply the above to $\frac{z}{|z|}$. Then the square roots of z are given by

$$\sqrt{z} = \pm \frac{1+\frac{z}{|z|}}{\left|1+\frac{z}{|z|}\right|}\sqrt{|z|}.$$

We now turn our attention to functions of a complex variable $f : \mathbb{C} \to \mathbb{C}$. A prime example is given by complex power series:

$$\sum_{n=0}^{\infty} a_n z^n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n z^n.$$

To find out when this limit exists we check when the sequence of partial sums is Cauchy. Take $M < N$ and compute:

$$\left|\sum_{n=0}^{N} a_n z^n - \sum_{n=0}^{M} a_n z^n\right| = \left|\sum_{n=M+1}^{N} a_n z^n\right| \leq \sum_{n=M+1}^{N} |a_n z^n| = \sum_{n=M+1}^{N} |a_n|\, r^n,$$

where $r = |z|$. This implies that if $\sum_{n=0}^{\infty} |a_n| r^n$ converges in $\mathbb{R}$, then $\sum_{n=0}^{\infty} a_n z^n$ converges in $\mathbb{C}$. Next, $\sum_{n=0}^{\infty} |a_n| r^n < \infty$ holds if there exists $\tilde{r} > r$ with $\sup_n |a_n| \tilde{r}^n < \infty$ because

$$\sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} a_n \tilde{r}^n \left(\frac{r}{\tilde{r}}\right)^n \leq \left(\sup_n |a_n| \tilde{r}^n\right) \sum_{n=0}^{\infty} \left(\frac{r}{\tilde{r}}\right)^n < \infty.$$

Definition 1.4. The *convergence radius* of a power series $\sum_{n=0}^{\infty} a_n z^n$ is defined as

$$R := \sup\{\tilde{r} \,:\, \sup_n |a_n| \tilde{r}^n < \infty\}.$$

- For $z \in D_R(0) = \{z \,:\, |z| < R\}$, the sum $\sum_{n=0}^{\infty} a_n z^n$ converges.
- For $|z| > R$, the sum $\sum_{n=0}^{\infty} a_n z^n$ diverges.

- For $|z| = R$ both convergence and divergence are possible.

Examples 1.5. The *exponential series*

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

has convergence radius $R = \infty$. The same holds for

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n},$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

These combine to give the Euler formula,

$$e^{iz} = \cos(z) + i\sin(z).$$

For φ real, $e^{i\varphi}$ lies on the unit circle. Let us see how to derive this from the definition. For n large we have

$$e^{i\varphi} = \left(e^{i\frac{\varphi}{n}}\right)^n \sim \left(1 + i\frac{\varphi}{n}\right)^n,$$

$$\left|1 + i\frac{\varphi}{n}\right| \leq 1 + \left(\frac{\varphi}{n}\right)^2.$$

Notice that

$$\left(1 + \frac{\varphi^2}{n^2}\right)^{n^2} \overset{n\to\infty}{\longrightarrow} e^{\varphi^2},$$

so

$$|e^{i\varphi}| \sim \left|1 + i\frac{\varphi}{n}\right|^n \sim \sqrt[n]{e^{\varphi^2}} \to 1.$$

Remark 1.6. General polynomials in x, y on $\mathbb{R}^2$ are of the form

$$\sum_{n,m=0}^{N} a_{n,m} x^n y^m = \sum_{n,m=0}^{N} a_{n,m} \left(\frac{z+\overline{z}}{2}\right)^n \left(\frac{z-\overline{z}}{2i}\right)^m = \sum_{n,m=0}^{N} b_{n,m} z^n \overline{z}^m.$$

In complex analysis we only consider the case $b_{n,m} = 0$ for $m \neq 0$.

Definition 1.7. A function $f : \mathbb{C} \to \mathbb{C}$ is called *complex differentiable* at $z \in \mathbb{C}$ if for $h \in \mathbb{C}$ with $|h|$ small enough we have

$$f(z+h) = f(z) + hg(z) + o(h) \tag{1.2}$$

such that for all $\varepsilon > 0$ there exists $\delta > 0$ with $|o(h)| < \varepsilon|h|$ for all $|h| < \delta$.

Theorem 1.8. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has convergence radius R, then*

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$$

also has convergence radius R and for $|z| < R$, f is complex differentiable at z.

Proof. We already know how to differentiate power series from real analysis. The proof of this theorem works exactly the same way as in the real case:

$$\begin{aligned} f(z+h) &= \sum_{n=0}^{\infty} a_n (z+h)^n = \sum_{n=0}^{\infty} \left(a_n z^n + n h z^{n-1} + \sum_{k=2}^{n} a_n \binom{n}{k} h^k z^{n-k} \right) \\ &= f(z) + h g(z) + o(h) \end{aligned}$$

and

$$\left| \frac{o(h)}{h} \right| \le |h| \sum_{n=0}^{\infty} |a_n| n^2 \sum_{k=0}^{n-2} \binom{n+2}{k} |h|^k |z|^{n+2-k} \le |h| \sum_{n=0}^{\infty} |a_n| n^2 (|z| + |h|)^{n+2}.$$

$\square$

Compare this to the real Taylor series in $\mathbb{R}^2$: let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be totally differentiable in z, then there exists a matrix A with

$$f(z+h) = f(z) + Ah + o(h) \tag{1.3}$$

and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|o(h)| \le \varepsilon |h|$ for $|h| < \delta$. Note that the product in (1.3) is the matrix product and the product in (1.2) is the product of complex numbers. They coincide if and only if

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Thus we find that a function $f(z) = (u(x,y), v(x,y))$ that is (real) totally differentiable at z is complex differentiable at z if and only if

$$\frac{\partial u}{\partial x}(z) = \frac{\partial v}{\partial y}(z) \quad \text{and} \quad \frac{\partial u}{\partial y}(z) = -\frac{\partial v}{\partial x}(z). \tag{1.4}$$

These are called the *Cauchy-Riemann differential equations.*

End of lecture 1. April 11, 2016