NONLINEAR DISPERSIVE EQUATIONS, SUMMER TERM 2013

HERBERT KOCH

1. Introduction

Nonlinearly interacting waves are often described by asymptotic equations. The derivation typically involves an Ansatz for an approximate solution where higher order terms - the precise meaning of higher order term depends on the context and the relevant scales - are neglected. Often a Taylor expansion of a Fourier multiplier is part of that process.

There is an immediate consequence: This type of derivation leads to a huge set of asymptotic equations, and one should search for a general understanding of interacting nonlinear waves by asking for precise results for specific equations.

The most basic asymptotic equation is probably the nonlinear Schrödinger equation, which describes wave trains or frequency envelopes close to a given frequency, and their self interactions. The Korteweg-de-Vries equation typically occurs as first nonlinear asymptotic equation when the prior linear asymptotic equation is the wave equation. It is one of the amazing facts that many generic asymptotic equations are integrable in the sense that there are many formulae for specific solutions, conserved quantities, Lax-Pairs and BiHamiltonian structures.

Recent progress in the study of dispersive equations involves

(1) Adapted function spaces. This idea goes back to Bourgain, Klainerman and Machedon, and Kenig, Ponce and Vega, it has been very successful, and it has been developed to an art.
(2) Harmonic analysis. Key words are stationary phase, Strichartz estimates, bilinear and multilinear estimates, Morawetz estimates and localization in phase space.
(3) Normal form analysis. There are at least two directions: The analysis of the flow near solitons and possible other special solutions, where the term normal form would refer to equations describing modes like position, velocity and scale of the soliton, and transformations leading to higher powers in the nonlinear terms which can often be handled easier.
(4) Minimal blow up solutions and induction on energy, settling the question of global existence for a number of defocusing equations, and also below the ground state mass for some focusing equations.

This text will focus on a contribution to adapted function spaces and their recent application to a number of dispersive equations.

1.1. Young's inequality and interpolation.

Young's inequality bounds convolutions in Lebesguespaces gives bounds for the convolution of two functions. It is part of the statement that the integral exists for almost all arguments of the convolution. Let m^d denote the d dimensional Lebesgue measure.

Lemma 1.1. *Let $1 \le p, q, r \le \infty$ satisfy*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2,$$

$$f \in L^p(\mathbb{R}^d), \quad g \in L^q(\mathbb{R}^n), \quad h \in L(\mathbb{R}^d).$$

Then

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x) g(x-y) h(y) dm^{2d}(x,y) \le \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.$$

We assume that the Lemma holds and choose $f(x) = e^{-|x|^2} \in L^r(\mathbb{R}^d)$. It follows by Fubini's theorem that $g(x-y)h(y)$ is integrable with respect to y for almost all x. The estimate of the lemma shows that

$$L^p(\mathbb{R}^d) \ni f \to \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} h(y) g(x-y) dm^d(y) \right) f(x) dm^d(x) \in \mathbb{R}$$

defines a linear form of norm $\le \|g\|_{L^q} \|g\|_{L^r}$ on L^r. Thus

$$\|g * h\|_{L^{p'}} \le \|g\|_{L^q} \|h\|_{L^r}$$

for

$$\frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p'}.$$

Proof of Lemma 1.1, as in [20]. Set

$$\frac{1}{\gamma_1} = 1 - \frac{1}{p}, \frac{1}{\gamma_2} = 1 - \frac{1}{q}, \frac{1}{\gamma_3} = 1 - \frac{1}{r}.$$

Then $1 \le \gamma_1 \le \infty$,

$$\frac{1}{\gamma_2} + \frac{1}{\gamma_3} = \frac{1}{p}, \frac{1}{\gamma_1} + \frac{1}{\gamma_3} = \frac{1}{q}, \frac{1}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r}$$

and

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} = 1.$$

Let

$$a(x,y) = |f(x)|^{p/\gamma_3} |g(x-y)|^{q/\gamma_3}, b(x,y) = |g(x-y)|^{q/\gamma_1} |h(y)|^{r/\gamma_1},$$
$$c(x,y) = |f(x)|^{p/\gamma_2} |h(y)|^{r/\gamma_2}.$$

Then

$$|f(x) g(x-y) h(y)| = a(x,y) b(x,y) c(x,y)$$

and, by applying Hölder's inequality twice

$$\int |f(x) g(x-y) h(y)| dm^{2d} \le \|a\|_{L^{\gamma_3}} \|b\|_{L^{\gamma_1}} \|c\|_{L^{\gamma_2}} = \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}.$$

□

There is an improvement: the weak Young inequality. Let (X, μ) be a measure space. We will often supress space and measure in the notation. The weak L^p spaces are defined by the quasinorm

$$\|f\|_{L^p_w} = \sup_{t>0} t \left(\mu(\{x : |f(x)| > t\}) \right)^{1/p}.$$

If $1 < p < \infty$ then there is an equivalent norm on L^p_w,

$$\|f\|_{L^p_w} \sim \sup_{t>0} t \left(\int_{\{x:|f(x)|>t\}} |f(y)| d\mu(y) \right)^{1/p}.$$

It is not hard to see the equivalence, and that the term on the right hand side defines a norm.

Proposition 1.2. *Suppose that*

$$1 < p, q, r < \infty, \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

$f \in L^p$ *and* $g \in L^q_w$. *Then* $f(x)g(x-y)$ *is integrable with respect to* x *for almost all* y *and*

$$\|f * g\|_{L^r} \le c_{p,q} \|f\|_{L^p} \|g\|_{L^q_w}.$$

This is a consequence of the Markinkiewicz interpolation theorem. We state and prove the following version.

Let X and Y be normed linear spaces. We denote by $L(X,Y)$ the normed space of bounded linear operators from X to Y.

Lemma 1.3 (Markinciewicz interpolation)**.** *Let* (X,μ) *and* (Y,ν) *be measure spaces. Let* $1 \le p_1 < p_2 \le \infty$, $1 \le q_1, q_2 \le \infty$, $q_1 \ne q_2$, $0 < \lambda < 1$,

$$\frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}, \qquad \frac{1}{q} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_2}.$$

Suppose that

$$T \in L(L^{p_1}(\mu), L^{q_1}_w(\nu)) \cap L(L^{p_2}(\mu), L^{q_2}_w(\nu)).$$

Then $T \in L(L^p_w(\mu), L^q_w(\nu))$, *and*

$$\|T\|_{L(L^p_w(\mu),L^q_w(\nu))} \le c \|T\|^{\lambda}_{L(L^{p_1}(\mu),L^{q_1}_w(\nu))} \|T\|^{1-\lambda}_{L(L^{p_2}(\mu),L^{q_2}_w(\nu))}$$

and, if $p \le q$, *then* $T \in L(L^p(\mu), L^q(\nu))$ *and*

$$\|T\|_{L(L^p(\mu),L^q(\nu))} \le c \|T\|^{\lambda}_{L(L^{p_1}(\mu),L^{q_1}_w(\nu))} \|T\|^{1-\lambda}_{L(L^{p_2}(\mu),L^{q_2}_w(\nu))}$$

with a constant c *depending only on the exponents.*

Proof of proposition 1.2 . Let $f \in L^p$ and $Tg : L^q \to L^r$ be the convolution with g. We interpolate the estimate with $p_1 = 1$ and $p_2 = p'$ and $q_1 = q$ and $q_2 = \infty$ to get the estimate in weak spaces

$$\|f * g\|_{L^r_w} \le \|g\|_{L^q_w} \|f\|_{L^p}.$$

Now we fix g and consider $T : f \to f * g$, and get

$$\|f * g\|_{L^r} \le c \|f\|_{L^p} \|g\|_{L^q_w}$$

by the second part of the Lemma. □

It is useful to generalize and sharpen the Markinciewiecz interpolation estimates before proving them.

Definition 1.4 (Lorentz spaces)**.** *Let* (A,μ) *be a measure space and* $1 \le p, q \le \infty$. *We define*

$$\|f\|_{L^{p,q}(\mu)} = \left(q \int_0^\infty \left(\mu(\{x : |f(x)| > t)^{1/p} t \right)^q \frac{dt}{t} \right)^{1/q}$$

with the obvious modification for $q = \infty$. *We denote by* $L^{pq}(\mu)$ *the set of all measurable functions* f *for which* $\|f\|_{L^{pq}(\mu)} < \infty$.

Properties:

(1) Since

$$\{x : |f(x) + g(x)| > t\} \subset \{x : |f(x)| > t/2\} \cup \{x : |g(x)| > t/2\}$$

it follows that

$$\mu(\{x : |f(x) + g(x)| > t\}) \le \mu(\{x : |f(x)| > t/2\}) + \mu(\{x : |g(x)| > t/2\})$$

and hence

$$\|f + g\|_{L^{pq}} \le c\,(\|f\|_{L^{pq}} + \|g\|_{L^{pq}})\,.$$

(2) For $q_1 \le q_2$

$$\|f\|_{L^{pq_2}} \le c\|f\|_{L^{pq_1}}.$$

We begin the proof with

$$\mu(\{|f| \ge t\})t^q = q\int_0^t \mu(\{|f| \ge t\})s^{q-1}ds \le q\int_0^t \mu(\{|f| \ge s\})s^{q-1}ds \le \|f\|^q_{L^{pq}}.$$

Now, if $q_1 < q_2$,

$$q_2\int_0^\infty [\mu(\{|f| \ge t\})^{1/p}t]^{q_2}\frac{dt}{t} \le \frac{q_2}{q_1}\|f\|^{q_2-q_1}_{L^{p,\infty}}\|f\|^{q_1}_{L^{p,q_1}} \le \frac{q_2}{q_1}\|f\|^{q_2}_{L^{p,q_1}}.$$

(3) If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ there exists $c > 0$ such that

$$\left|\int fgd\mu\right| \le c\|f\|_{L^{pq}}\|g\|_{L^{p'q'}}.$$

For the proof we define $f^* : (0,\infty) \to \mathbb{R}^+$ to be the unique function with

$$m^1(\{\tau : f^*(\tau) > t\}) = \mu(\{x : f(x) > t\})$$

for all $t > 0$. Then, using Fubini several times (with the Lebesgue measure $\mu = m^d$ for definiteness, but the argument holds for general measures)

$$\begin{aligned}\int |fg|dm^d =& m^{d+2}(\{(x,s,t) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} : 0 < s < |f(x)|, 0 < t < |g(x)|\})\\ =& \int_{\mathbb{R}^+\times\mathbb{R}^+} m^d(\{x : |f(x)| > s\} \cap \{x : |g(x)| > t\})dsdt\\ \le& \int_{\mathbb{R}^+\times\mathbb{R}^+} \min\{m^d(\{|f(x)| > s\}), m^d(\{|g(x)| > t\})\}dsdt\\ =& \int_{\mathbb{R}^+\times\mathbb{R}^+} m^1(\{|f^*(x)| > s\} \cap \{|^*g(y)| > t\})dsdt\\ =& \int_0^\infty f^*(\tau)g^*(\tau)d\tau\end{aligned}$$

which we use below,

$$\begin{aligned}\int fgd\mu \le& \int_0^\infty f^*(t)g^*(t)dt\\ =& \int_0^\infty (t^{1/p}f^*)(t^{1/p'}g^*(t))dt/t\\ \le& \left(\int_0^\infty t^{(q/p)-1}(f^*)^q dt\right)^{1/q}\left(\int_0^\infty t^{(q'/p')-1}(g^*)^{q'}dt\right)^{1/q'}\end{aligned}.$$

The last inequality is an application of Hölder's inequality. The proof of the third part is completed by the equality

$$\frac{q}{p}\int_0^\infty t^{(q/p)-1}(f^*(t))^q dt = q\int_0^\infty (\mu(|f(x)|>s))^{q/p}s^{q-1}ds. \tag{1.1}$$

in one dimensional calculus. We observe that

$$s \to m^1(\{\tau : f^*(\tau) > s\})$$

is the inverse of f^*. Both functions are monotonically decreasing.

Let f and f^{-1} be inverse nonnegative mononically decreasing functions, and g and h nonnegative monotonically increasing functions with antiderivatives G and H with

$$H(t)G\circ f(t) \to 0$$

as $t\to\infty$ and $t\to 0$. Then by an integration by parts and one substitution

$$\int_0^\infty hG\circ f dt == -\int_0^\infty Hg\circ f f' dt = \int_0^\infty H\circ f^{-1}(s)g(s)ds.$$

This specializes to (1.1). Moreover, checking the inequalities shows that

$$\|f\|_{L^{pq}} \le c\sup\{\int fgd\mu : \|g\|_{L^{p'q'}} \le 1\}.$$

(4) This pairing defines a duality isomorphism if $1<p<\infty$ and $1\le q<\infty$. In particular all spaces L^{pq} with $1<p$ are Banach spaces.

$$L^{p'q'} \ni g \to (f \to \int fgd\mu) \in (L^{pq})^*$$

To prove it we choose B to be a measurable set of positive finite measure. There exists $\tilde p > p$ so that $L^{\tilde p}(B) \subset L^{pq}$. If l is a bounded linear functional on L^{pq} then it defines a bounded linear functional on $L^{\tilde p}$ which is represented by a function $g \in L^{\tilde p'}(\mu)$. The previous step gives a bound for $\|g\chi_B\|_{L^{p'q'}}$ in terms of l.

We order the measurable subsets of A by inclusion up to sets of measure zero. This defines a partial order on the subsets on which the duality statement holds. Every chain has an upper bound, the union of the chain. By the lemma of Zorn there is a maximal element. The procedure above allows to show that the maximal set is necessarely the full space.

In particular duality allows to define an equivalent norm on $L^{pq}(\mu)$ for $1\le p\le\infty$ and $1\le q\le\infty$. Completeness of dual spaces is obvious. Completeness of $L^{p1}(\mu)$ is left as an exercise.

Lemma 1.5. *Suppose that* $1 \le p_1, p_2, q_1, q_2 \le \infty$,

$$T \in L(L^{p_1 1}(\mu), L^{q_1\infty}(\nu)) \cap L(L^{p_2 1}(\mu), L^{q_2\infty}(\nu)),$$

$p_1 \ne p_2$, $q_1 \ne q_2$, $0<\lambda<1$ *and*

$$\frac{1}{p} = \frac{1-\lambda}{p_1} + \frac{\lambda}{p_2}, \frac{1}{q} = \frac{1-\lambda}{q_1} + \frac{\lambda}{q_2}$$

and $1 \le r \le \infty$.

Then the operator can be continuously extended to $T \in L(L^{pr}(\mu), L^{qr}(\nu))$. *Moreover*

$$\|T\|_{L(L^{pr}(\mu),L^{qr}(\nu))} \le c\|T\|^{\lambda}_{L(L^{p_1}(\mu),L^{q_1}_w(\nu))}\|T\|^{1-\lambda}_{L(L^{p_2}(\mu),L^{q_2}_w(\nu))}.$$

Proof. An easy calculation shows

$$\frac{1-\frac{p}{p_2}}{1-\frac{p}{p_1}} = \frac{1-\lambda}{\lambda} \tag{1.2}$$

This will be useful lateron. Let $t > 0$ and

$$f_t(x) = \begin{cases} f(x) & \text{if } |f(x)| \le t \\ tf(x)/|f(x)| & \text{if } |f(x)| > t \end{cases}$$

and $f^t = f - f_t$. Then

$$f = f_t + f^t$$

and, if $p_1 < p < p_2$, which we assume in the sequel,

$$\|f^t\|_{L^{p_1}} \le (p-p_1)^{1/p_1} t^{1-\frac{p}{p_1}} \|f\|_{L^p_w}^{\frac{p}{p}1}$$

and

$$\|f_t\|_{L^{p_2}} \le (p_2-p)^{1/p_2} t^{1-\frac{p}{p_2}} \|f\|_{L^p_w}^{\frac{p}{p}2}$$

with obvious modifications if $p_2 = \infty$.

Moreover, by the triangle inequality,

$$\{|Tf| > t\} \subset \{Tf^s > t/2\} \cup \{Tf_s > t/2\}.$$

Let

$$a_1 = \|T\|_{L(L^{p_1}, L^{q_1}_w)} \qquad a_2 = \|T\|_{L(L^{p_2}, L^{q_2}_w)}$$

and

$$s = t^{\frac{q_2-q_1}{q_2(1-\frac{p}{p_2}) - q_1(1-\frac{p}{p_1})}} a_1^{\frac{(1-\lambda)q/q_1-1}{1-p/p_1}} a_2^{\frac{\lambda q/q_2-1}{1-p/p_2}}.$$

Step 1. The bound in weak L^p space. We want to prove

$$\lambda \nu(\{|Tf(x)| > t\})^{1/q} \le c a_1^{1-\lambda} a_2^{\lambda}$$

for $\|f\|_{L^p_w} = 1$ with c depending only on the exponents. Then

$$\begin{aligned}
\lambda^q \mu(\{|Tf| > t\}) \le& c\left(t^{q-q_1}\|Tf^s\|^{q_1}_{L^{q_1}_w} + t^{q-q_2}\|Tf_s\|^{q_2}_{L^{q_2}_w}\right)\\
\le& c\left(t^{q-q_1}a_1^{q_1}\|f^s\|^{q_1}_{L^{p_1}} + t^{q-q_2}a_2^{q_2}\|f_s\|^{q_2}_{L^{p_2}}\right)\\
=& c\left(t^{q-q_1}s^{q_1-q_1p/p_1}\|f\|^{pq_1/p_1}_{L^p_w} + t^{q-q_2}s^{q_2-q_2p/p_2}\|f\|^{pq_2/p_2}_{L^p_w}\right)\\
=& c\left(t^{q-q_1-\frac{q_1(q_2-q_1)}{q_2\frac{1-\lambda}{\lambda}+q_1}} + t^{q-q_2-\frac{q_2(q_1-q_2)}{q_1\frac{\lambda}{1-\lambda}+q_2}}\right)a_1^{q(1-\lambda)}a_2^{q\lambda}\\
=& c\left(t^{q_1[q/q_1-1-(q/q_1-q/q_2)\lambda]} + t^{q_2[q/q_2-1-(q/q_2-q/q_1)(1-\lambda)]}\right)a_1^{q(1-\lambda)}a_2^{q\lambda}\\
=& c a_1^{q(1-\lambda)}a_2^{q\lambda}.
\end{aligned}$$

This completes the proof of the weak type estimate.

Step 2: The endpoint $L(L^{p1}, L^{q1})$ We assume that $1 < p_1, p_2, q_1, q_2 < \infty$ which can be achieved by the first step.

By duality, with constant changing from line to line

$$\begin{aligned}\|Tf\|_{L^{qr}} \le & c\sup\{\int (Tf)gd\nu : \|g\|_{L^{q'r'}}\\ =& c\sup\{\int fT^*gd\nu : \|g\|_{L^{q'r'}} \le 1\}\\ =& c\|f\|_{L^{pq}}\|T^*\|_{L(L^{q',r'}(\nu),L^{p',q'}(\mu))}\end{aligned}$$

and hence, for $1 < p < \infty$,

$$\|T\|_{L(L^{pr},L^{qr})} \le c\|T^*\|_{L(L^{q'r'},L^{p'r'})}.$$

We apply this with $L^{p_1 1} \to L^{q_1\infty}$ to see that

$$\|T^*\|_{L(L^{q_i'1},L^{p_i'\infty})} \le c\|T\|_{L(L^{p_i1},L^{q_i\infty})}$$

for $i = 1, 2$. From Step 1

$$\|T^*\|_{L(L^{q'\infty},L^{p'\infty})}$$

satisfies the desired bounds. Duality again gives the statement for $r = 1$.

Step 3: Interpolation in L^p.

Suppose that $T \in L(L^1(\mu), L^1(\nu)) \cap L(L^\infty(\mu), L^\infty(\mu))$ with norm $\le \frac{1}{2}$. Then

$$\|Tf\|_{L^p(\nu)} \le \left(\frac{1}{p-1}\right)^{1/p} \|f\|_{L^p(\mu)}$$

We begin the proof with the observation

$$\{|Tf| > t\} \subset \{Tf_t > t/2\} \cup \{Tf^t > t/2\}.$$

The first set is empty by assumption on the norm of T. Hence

$$\begin{aligned}p\int \nu(\{|Tf| > t\})t^{p-1}dt \le & p\int \nu(\{Tf^t > t/2)t^{p-1}dt\\ \le & p\int_0^\infty \|f^t\|_{L^1}t^{p-2}dt\\ =& p\int_0^\infty\int_t^\infty \mu(\{|f| \ge s\})ds t^{p-2}dt\\ =& p\int_0^\infty\int_0^s t^{p-2}dt\mu(\{|f| \ge s\})ds\\ =& \frac{1}{p-1}\|f\|^p_{L^p}\end{aligned}$$

Step4: Conclusion

We have proven the bounds for $\|T\|_{L(L^{p,\infty},L^{q,\infty})}$ and $\|T\|_{L(L^{p,1},L^{q,1})}$

Let

$$f_t(x) = \begin{cases} f(x) & \text{if } (\mu(|f(y)| > |f(x)|)^{1/p}|f(x)| \le t\\ 0 & \text{otherwise}\end{cases}$$

and $f^t = f - f_t$. We assume that the bounds for T are $1/2$ as above. Since

$$\|f_t\|_{L^{p,\infty}} \le t$$

we have

$$\{|Tf(x)| \ge t\} \subset \{|Tf^t(x)| \ge t/2\}$$

Let

$$g^t(s) = \mu(\{|f^t| > s\})^{1/p}s \le \mu(\{|f| > s\})^{1/p}s$$

We proceed as in Step 3.

□

1.2. **Complex interpolation: The theorem of Riesz-Thorin.** The Riesz-Thorin interpolation theorem states the following.

Theorem 1.6. *Let* $1 \le p_1, p_2, q_1, q_2 \le \infty$. *Let* T_λ, $0 \le \operatorname{Re}\lambda \le 1$ *be an operator from* $L^1 \cap L^\infty \to L^1 + L^\infty$. *Suppose that*

$$\lambda \to \int T_\lambda f g$$

is continuous in $0 \le \operatorname{Re}\lambda \le 1$, *holomorphic inside the strip, for all* $f \in L^1 \cap L^\infty$ *and* $g \in L^1 \cap L^\infty$. *Suppose that*

$$\sup_{\operatorname{Re}\lambda = 0} \|T_\lambda\|_{L(L^{p_0}, L^{q_0})} = C_0$$

and

$$\sup_{\operatorname{Re}\lambda = 1} \|T_\lambda\|_{L(L^{p_1}, L^{q_1})} = C_1.$$

Then

$$\|T_\lambda\|_{L(L^p, L^q)} \le C_0^{1-\operatorname{Re}\lambda} C_1^{\operatorname{Re}\lambda}$$

if

$$\frac{1-\operatorname{Re}\lambda}{p_0} + \frac{\operatorname{Re}\lambda}{p_1} = \frac{1}{p} \qquad \frac{1-\operatorname{Re}\lambda}{q_0} + \frac{\operatorname{Re}\lambda}{q_1} = \frac{1}{q}$$

The proof relies on the three lines theorem in complex analysis:

Lemma 1.7 (Three lines theorem)**.** *Suppose that* v *is a bounded holomorphic function on the strip* $C = \{z = x + iy : 0 < x < 1\}$ *and that it is continuous on the closure. Then*

$$|v(x)| \le (\sup_y |v(iy)|)^{1-x} (\sup_y |v(1+iy)|)^x.$$

Proof. By the maximum principle of harmonic functions any harmonic function on a bounded open set, which is continuous on the closure, assumes the maximum of the modulus at the boundary. This is true for

$$u_\varepsilon = e^{\varepsilon(x+iy)^2} \sup_y |u(1+iy)|)^{-z} \sup_y |u(iy)|)^{z-1}$$

on $C \cap \overline{B_R(0)}$ for every R. This function tends to 0 as $y \to \infty$ hence

$$|u_\varepsilon(x+iy)| \le \max\{\sup_y |u(iy)|)^{1-x}, \sup_y |u(1+iy)|)^x\}$$

and $\varepsilon \to 0$ gives the result. □

Proof of Theorem 1.6. Let $f \in L^1(\mu) \cap L^\infty(\mu)$ and $g \in L^1(\nu) \cap L^\infty(\nu)$. Then, by assumption

$$v(\lambda) = \int T_\lambda f g d\nu$$

is a bounded analytic function. By the three lines theorem 1.7 we have

$$|v(\lambda)| \le \sup_t \max\{|v(it)|, |v(1+it)|\}$$

and

$$\left| \int T_{it} f g d\nu \right| \le \|T_{it} f\|_{L^{q_0}} \|g\|_{L^{q_0'}} \le C_0 \|f\|_{L^{p_0}} \|g\|_{L^{q_0'}}.$$

Similarly

$$\left|\int T_{1+it} f g d\nu\right| \le \|T_{1+it} f\|_{L^{q_1}} \|g\|_{L^{q_1'}} \le C_0 \|f\|_{L^{p_1}} \|g\|_{L^{q_1'}},$$

thus

$$\left|\int (T_\lambda f) g d\mu\right| \le \max\{C_0, C_1\} \left(\|f\|_{L^{p_0}} \|g\|_{L^{q_0'}} + \|f\|_{L^{p_1}} \|g\|_{L^{q_1'}} \right)$$

and we could derive that

$$\|T\|_{L(L^{p_0} \cap L^{p_1}, L^{q_0} + L^{q_1})} \le \max\{C_0, C_1\}$$

but we will avoid this step. Let $f \in L^p$ and $g \in L^{q'}$. We want to prove

$$\left|\int g T_\lambda f\right| \le \|f\|_{L^p} \|g\|_{L^{q'}} \sup_y \|T_{iy}\|^{1-\lambda}_{L(L^{p_1}, L^{q_1})} \sup_y \|T_{1+iy}\|^{\lambda}_{L(L^{p_2}, L^{q_2})}. \tag{1.3}$$

for $f \in L^p$ and $g \in L^{q'}$. The theorem follows then by an duality argument. Moreover it suffices to consider a dense set of functions, which are measurabe, bounded, and for which there is $\varepsilon > 0$ such that either the functions vanish at a point, or else are at least of size ε. Moreover we may restrict to f and g with $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$.

Let

$$f_z(x) = f(x)/|f(x)| |f(x)|^{(1-z)\frac{p}{p}_0 + z\frac{p}{p}_1},$$

$$g_z(x) = g(x)/|g(x)| |g(x)|^{(1-z)\frac{q'}{q_0'} + z\frac{q'}{q_1'}}$$

and

$$v(z) = \int g_z(y) T_z f_z(y) d\nu(y).$$

This is a bounded holomorphic map from the strip to $L^1 \cap L^\infty$ with values in $\mathbb{C}$. We claim that it is continuous on the closure of the strip at an arbitrary point λ. We write

$$v(z) - v(\lambda) = \int g_\lambda (T_z - T_\lambda) f_\lambda d\nu + \int (g_z - g_\lambda) T_z f_\lambda + g_z T_z (f_z - f_\lambda) d\nu.$$

The first term tends to zero as $z \to \lambda$ by assumption. Then

$$g_z - g_\lambda \to 0 \quad \text{ and } g_z - f_\lambda \to 0 \quad \text{ as } z \to \lambda$$

in $L^1 \cap L^\infty$. Continuity follows by the uniform bound above.

We turn to complex differentiability at an abitrary point λ in the interior. Indeed

$$\frac{v(z) - v(\lambda)}{z - \lambda} = \frac{\int g_\lambda (T_z - T_\lambda) f_\lambda d\nu}{z_\lambda} + \int \frac{g_z - g_\lambda}{z - \lambda} T_z f_\lambda d\nu + \int g_z T_z \frac{f_z - f_\lambda}{z - \lambda} d\nu$$

The first term converges to a complex number by assumption. Moreover

$$\frac{g_z - g_\lambda}{z - \lambda}$$

converges to a function g'_λ in $L^1 \cap L^\infty$ as $z \to \lambda$. Let $\tilde g$ be the difference between the difference quotient and g'_λ. Then

$$\int \frac{g_z - g_\lambda}{z - \lambda} T_z f_\lambda d\nu = \int g' T_\lambda f_\lambda d\nu + \int \tilde g T_z f_\lambda d\nu + \int g' (T_z - T_\lambda) f_\lambda d\nu.$$

The second term tends to zero since $\tilde g$ tends to zero in $L^1 \cap L^\infty$ and the third one by the continuity assumption as $z \to \lambda$. Similarly we deal with the last term.

We turn to the behaviour at the boundary.

$$|v(it)| = \int T_{it} f_{it} g_{it} d\nu \le \|T_{it}\|_{L(L^{p_0},L^{p_1})} \|f_{it}\|_{L^{p_0}} \|g_{it}\|_{L^{q_0}}$$

and

$$\|f\|_{L^{p_0}} = \|f\|_{L^p}^{p_0/p} = 1 = \|g_{it}\|_{L^{q_0'}} = \|g\|_{L^{q'}}^{q_0'/q'}.$$

We apply the three lines theorem 1.7 to get

$$|v(z)| \le \sup_y \|T_{iy}\|^{1-x}_{L(L^{p_1},L^{q_1})} \sup_y \|T_{1+iy}\|^x_{L(L^{p_2};L^{q_2})}$$

We evaluate it at $z = \lambda$, which gives inequality (1.3). □

1.3. **Stationary phase and dispersive estimates.** We begin with a number of evaluations of integrals. Let

$$I_d = \int_{\mathbb{R}^d} e^{-|x|^2} dm^d(x).$$

Then, with Fubini,

$$\begin{aligned} I_{d_1+d_2} =& \int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} e^{-|x_1|^2-|x_2|^2} dm^{d_1+d_2}(x) \\ =& \int_{\mathbb{R}^{d_1+d_2}} e^{-|x_1|^2} e^{-|x_2|^2} dm^{d_1+d_2}(x) \\ =& \int_{\mathbb{R}^{d_1}} e^{-|x_1|^2} \int_{\mathbb{R}^{d_2}} e^{-|x_2|^2} dm^{d_2} dm^{d_1} \\ =& I_{d_2} \int_{\mathbb{R}^{d_1}} e^{-|x_1|^2} dm^{d_1} \\ =& I_{d_1} I_{d_2} \end{aligned}$$

hence

$$I_d = I_1^d.$$

Applying Fubini twice we get

$$\begin{aligned} I_d =& m^{d+1}(\{(x,t) : 0 < t < e^{-|x|^2}\}) \\ =& \int_0^1 m^d(\{x : e^{-|x|^2} > t\}) dt \\ =& \int_0^1 m^d(B_{(-\ln(t))^{1/2}}(0)) dt \\ =& m^d(B_1(0)) \int_0^1 (-\ln(t))^{d/2} dt \\ =& m^d(B_1(0)) \int_0^\infty s^{d/2} e^{-s} ds \\ =& m^d(B_1(0)) \Gamma(\frac{d}{2}+1) \end{aligned}$$

and hence $I_2 = \pi$, $I_1 = \sqrt{\pi}$, $I_d = \pi^{d/2}$,

$$m^d(B_1(0)) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$$

and

$$\Gamma(\frac{1}{2}) = 2\Gamma(\frac{3}{2}) = \sqrt{\pi}.$$

We proceed with

$$I(\tau) := \int_{-\infty}^{\infty} e^{-\frac{\tau}{2}x^2} dx$$

for $\operatorname{Re}\tau > 0$. Then

$$\begin{aligned} \frac{d}{dt}\sqrt{t+is}I(t+is) =& \frac{1}{2(t+is)}\sqrt{t+is}I(t+is) - \frac{1}{2}\sqrt{t+is}\int_{-\infty}^{\infty} e^{-\frac{t+is}{2}x^2} x^2 dx \\ =& \frac{\sqrt{t+is}}{2(t+is)}\left(I(t+is) + \int_{-\infty}^{\infty} \frac{d}{dx} e^{-\frac{t+is}{2}} x dx\right) \\ =& \frac{\sqrt{t+is}}{2(t+is)}\left(I(t+is) - \int_{-\infty}^{\infty} e^{-\frac{t+is}{2}x^2} dx\right) \\ =& 0 \end{aligned}$$

and similarly

$$\frac{d}{ds}\sqrt{t+is}I(t+is) = 0$$

Thus

$$\sqrt{\tau}I(\tau) = \sqrt{2}I(2) = \sqrt{2\pi}$$

and hence

$$\int e^{-\frac{\tau}{2}x^2} dx = \sqrt{\frac{2\pi}{\tau}}. \tag{1.4}$$

Now we fix τ and study

$$\int e^{-\frac{\tau}{2}x^2} x^k dx.$$

This vanishes when k is odd, since then the integrand is an odd function. Let

$$\begin{aligned} J(k) = \int e^{-\frac{\tau}{2}x^2} x^{2k} dx = & \frac{2k-1}{\tau} J(k-1) \\ =& 1 * 3 * \cdots * (2k-1)\tau^{-k}\sqrt{\frac{2\pi}{\tau}} \\ =& \frac{1}{2^k k!}(\tau^{-1}\frac{d^2}{dx})^k x^{2k}\Big|_{x=0}\sqrt{\frac{2\pi}{\tau}}. \end{aligned}$$

Let p be a polynomial. It is a sum of monomials and hence

$$\int e^{-\frac{\tau}{2}x^2} p(x) dx = \sqrt{\frac{2\pi}{\tau}} \sum_{k=0}^{\infty} \frac{1}{2^k k!}(\tau^{-1}\frac{d^2}{dx})^k p(x)\Big|_{x=0}$$

The higher dimensional case is contained in the following lemma. Let $A = A_0 + iA_1$ be a real symmetric $d \times d$ matrix with A_0 positive definite. This is equivalent to all eigenvalues λ_j being in $\{\lambda : \operatorname{Re}\lambda > 0\}$. Let (a_{ij}) be the inverse. By an abuse of notation we set

$$\det(A)^{-1/2} = \prod \lambda_j^{-1/2}.$$

Lemma 1.8. *Let p be a polynomial. Then*

$$\int e^{-\frac{1}{2}x^TAx}p(x)\,dx = (2\pi)^{-d/2}(\det A)^{-1/2}\frac{1}{2^k k!}(\sum_{i,j=1}^d a_{ij}\partial_{ij}^2)^k p(x)\Big|_{x=0}. \tag{1.5}$$

Proof. We begin with a fact from linear algebra and claim that there exists a real $d\times d$ matrix B and a diagonal matrix D such that

$$A = BDB^T.$$

By the Schur decomposition there is an orthogonal matrix O and a diagonal matrix with nonnegative entries such that

$$A_0 = OD_0O^T.$$

We set $B_0 = O\sqrt{D}$. Then

$$A_0 + iA_1 = B_0(1 + iB_0^{-1}A_1B_0^{-T})B_0^T$$

Again by the Schur decomposition there is an orthogonal matrix U with and a diagonal matrix D_1 with

$$B_0^{-1}A_1B_0^{-T} = UD_1U^T$$

hence

$$A_0 + iA_1 = B(1 + iD_1)B^T$$

with $B = B_0U$. We set $D = 1 + iD_1$.

We change coordinates to $y = B^Tx$. Then

$$\int e^{-\frac{x^TAx}{2}}p(x)dm^d(x) = (\det B)^{-1}\int e^{-\frac{y^T(1+iD_1)y}{2}}p(B^{-T}y)dm^d(y)$$

and by Fubini and the previous calculations

$$\int e^{-\frac{y^T(1+iD_1)y}{2}}y^\alpha dm^d(y) = 0$$

if one of the indices is odd, and otherwise, with d_j the diagonal entries of D_1,

$$\begin{aligned}\int e^{-\frac{y^TDy}{2}}y^{2\alpha}dm^d(y) =&(2\pi)^{d/2}\det(D)^{-1/2}\frac{1}{2^{|\alpha|}\alpha!}\prod((1+id_j)^{-1}\partial^2_{y_jy_j})^{\alpha_j}y_j^{2\alpha_j}\Big|_{y=0}\\ =&(2\pi)^{d/2}\det(D)^{-1/2}\frac{1}{2^{|\alpha|}|\alpha!|}[D^{-1}\partial^2]^{|\alpha|}y^\alpha\Big|_{y=0}\end{aligned}$$

Thus

$$\int e^{-\frac{y^TDy}{2}}q(y)dm^d(y) = (2\pi)^{d/2}\det(D)^{-1/2}\sum_{k=0}^\infty[D^{-1}\partial^2]^kq(y)\Big|_{y=0}$$

We complete the calculation by

$$(\det A)^{1/2} = (\det D)^{1/2}(\det B)$$

and, by the chain rule,

$$\sum a_{ij}\partial^2_{x_ix_j}p(x) = D^{-1}\partial_y^2p(B^{-T}y).$$

□

Observe that the formulas on the right hand side have a limit as A tends to a purely imaginary invertible matrix. We call the integral on the left hand side oscillatory integral in that limit.

Oscillatory integrals play a crucial role when studying dispersive equations. We consider

$$I = \int_{\mathbb{R}^s} a(\xi) e^{i\tau\phi(\xi)} d\xi.$$

where a and ϕ are smooth functions. The simplest result is

Lemma 1.9. *Suppose that $a \in C_0^\infty(\mathbb{R}^d)$, $\phi \in C^\infty(\mathbb{R}^d)$ with $\operatorname{Im}\phi \geq 0$ and*

$$|\nabla\phi| + \operatorname{Im}\phi > 0$$

on supp a. Given $N > 0$ there exists c_N with

$$|I(\tau)| \leq c_N \tau^{-N}.$$

The constant c depends only on N, the lower bound above, and derivatives up to order N.

Proof. By compactness there is $\kappa > 0$ such that

$$|\nabla\phi| + \operatorname{Im}\phi > \kappa$$

on supp $a(\xi)$. Using a partition of unity we may restrict to the two cases:

(1) $\operatorname{Im}\phi > \kappa/2$ on supp a, in which case we get a bound $Ce^{-\kappa\tau/2}$
(2) $|\nabla\phi| \geq \kappa/2$ on supp a, which we consider now.

We write

$$\begin{aligned} \int a(\xi) e^{i\tau\phi(\xi)} d\xi =& (i\tau)^{-1} \int a(\xi) |\nabla\phi|^{-2} \nabla\phi \nabla e^{i\tau\phi(\xi)} d\xi \\ =& (i\tau)^{-1} \int (\nabla \cdot (\frac{a(\xi)\nabla\phi}{|\nabla\phi|^2})) e^{i\tau\phi(\xi)} d\xi. \end{aligned}$$

Induction implies the full statement. □

In many cases these bounds hold even for non compactly supported a.

Lemma 1.10. *Suppose that $A = A_0 + iA_1$ be invertible with A_0 positive semi-definite. Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be identically 1 in a ball of radius 1, and supported in $B_2(0)$, and let a a smooth function with uniformly bounded derivatives of order M for some $M > 0$ and $0 < s < \frac{1}{2}$. Then*

$$\left| \int e^{-\frac{\tau}{2} x^T A x} e^{-\varepsilon |x|^2} a(x)(1 - \eta(x\tau^{-s})) \right| \leq c_N \tau^{-N}$$

with c_N depending only on N, the norm of A and its inverse, and derivatives up to some order $M(N)$ of a and η, but not on $\varepsilon > 0$. The limit $\varepsilon \to 0$ exists.

We will use the formula with $\varepsilon = 0$.

Proof. We argue similarly to above. Each integration by parts gains as a factor τ, unless the derivative falls upon η. In that case the gain is only τ^{1-s} and we also loose a power of $|x|^{-1}$. Otherwise we get a factor $|x|^{-1} + |x|^{-2}$.

On the support

$$\tau^{-1}|x|^{-2} + \tau^{s-1}|x|^{-1} \leq c\tau^{2s-1}.$$

and integrations by parts gain us τ^{2s-1}. If no derivatives fall on η there remains an integration over whole space, but we gain a factor bounded by constant times

$|x|^{-1} + |x|^{-2}$ for each integration by parts. After K integrations by parts we can bound the integrand (if no derivatives fall on η) by

$$\tau^{K(s-1)}(1+|x|)^M |x|^{-K}$$

which for $K > M + d$ gives

$$\begin{aligned}\left| \int e^{-\tau \frac{1}{2} x^t A x} e^{-\varepsilon |x|^2} a(x)(1-\eta)(x\tau^{-s}) \right| \le & c_0 \tau^{(K(s-1)} \int_{|x| \ge \tau^{-s}} (1+|x|)^M |x|^{-K} \\ \le & c_1 \tau^{K(s-1)+s(K-d)} \end{aligned}$$

and we gain integrability uniform in ε. If a derivative falls on $e^{-\varepsilon|x|^2}$ this gives

$$-2x_j \varepsilon e^{-\varepsilon |x|^2}$$

which is bounded by a constant times $\min\{\sqrt{\varepsilon}, |x|^{-1}\}$. The integrand (after the integrations by parts) converges pointwise with a majorant as above. This gives the limit as $\varepsilon \to 0$. □

Similar statements hold for more general phase functions if

$$|\nabla \phi| \ge c|x|^\delta \qquad \text{for } |x| \ge R$$

and

$$|\partial^\alpha \phi| \le |x|^{-\delta} |\nabla \phi| \qquad \text{for } |x| \ge R$$

some R and δ, and $|\alpha| \ge 2$.

Lemma 1.11. *Let A be invertible, symmetric, with real part positive semidefinite, and $a \in C^\infty$ with bounded derivatives of order $\ge M$. Then*

(1.6)

$$\left| \int e^{-\frac{\tau}{2} x^T A x} a(x) dx - (2\pi)^{d/2} \tau^{-d/2} (\det A)^{-1/2} \sum_{k=0}^N \tau^{-k} (\sum_{ij} a_{ij} \partial^2)^k a \Big|_{x=0} \right| \le c_N \tau^{-N-1-d/2}.$$

Proof. We subtract the Taylor expansion of a up to some order. We choose $0 < s < \frac{1}{2}$ decompose the integral into

$$\int e^{-\frac{1}{2} x^T A x} \eta(x\tau^{-s})(a(x) - p(x)) dx + \int e^{-\frac{1}{2} x^T A x} [1 - \eta(x\tau^{-s})](a(x) - p(x)) dx.$$

The first integral is small by Lemma 1.10, and the second is bounded by

$$\tau^{s(d+N-M)}$$

where N is the degree of the Taylorpolynomial, and M is the integer for which the derivatives are bounded. □

Now we consider

$$I(\tau) = \int e^{i\tau \phi(x)} a(x) dx$$

where a is compactly supported, 0 is the only point in the support where the imaginary part of ϕ and $\nabla \phi$ vanish, the imaginary part of ϕ is nonnegative.

Lemma 1.12. *Let $\frac{1}{3} < s < \frac{1}{2}$. Then, with η and η as above, $a \in C_0^\infty$ and $N > 0$*

$$\left| \int e^{-\tau \phi(x)} \eta(x\tau^{-s}) a(x) dx \right| \le c_N \tau^{-N}.$$

Proof. The proof is the same as for the quadratic phase. Again this formula often holds without assuming compact support. □

We write

$$\phi(x) = a_0 + \frac{i}{2} x^T A x + \psi(x)$$

where ψ is smooth with $\psi(x) = O(|x|^3)$.

Theorem 1.13 (Stationary phase). *Under the assumptions above*

$$\left| \int e^{i\tau\phi} a(x) dx - (2\pi)^{d/2} \tau^{-d/2} (\det A)^{-1/2} e^{\phi(0)} \sum_{k=0}^{N} \frac{1}{2^k k! \tau^k} (a_{ij}\partial^2)^k [e^{i\tau\psi(x)} a(x)]_{x=0} \right| \le c\tau^{-d/2-N+1/3}.$$

Proof. We assume that the real part of A is positive definite. The general statement follows then by an obvious limit.

We choose M large and write $e^{i\tau\psi} a = p_M(x) + r_M(x)$ where p_M is the Taylor polynomial of degree M, and r_M is the remainder term. Clearly p_M depends on τ with typical terms of the type being polynomials in τx^α where α is a multiindex of length at least 3, and x_j. We write the term in the bracket as a sum of three terms,

$$\int e^{i\tau\phi} a(x)(1 - \eta(x\tau^s)) dx$$

$$\int e^{-\frac{\tau}{2} x^T A x} p_M(x)(1 - \eta(x\tau^s)) dx$$

and

$$\int \eta(x\tau^s) \left[e^{i\tau\phi} a(x) - e^{-\frac{\tau}{2} x^T A x} p_M(x) \right].$$

Lemma 1.10 and Lemma 1.12 control the first and the second term.

The third term is bounded by

$$\tau^{-ds+M(1-2s)}.$$

We choose s between $\frac{1}{3}$ and $\frac{1}{2}$ and M large. Finally we check the bound for the sum from $N+1$ to M term by term using Lemma 1.8. □

In the one dimensional setting the situation the Lemma of van der Corput provides an extremely useful and simple estimate.

Lemma 1.14. *Suppose that $d = 1$, a is of bounded variation with support in $[c, d]$, $\phi \in C^k(\mathbb{R})$ with $k \ge 1$, ϕ real, and $\phi^{(k)}(\xi) \ge \tau$ for $\xi \in [c, d]$ and supp $a \subset [c, d]$. If $k = 1$ we assume in addition that $\operatorname{Re} \phi'$ is monotone. Then*

$$I = \left| \int a(x) e^{i\phi(x)} dx \right| \le 3k\tau^{-1/k} \int |a'| dx.$$

Proof. We begin with $k = 1$, assuming that ϕ' is monotone. It suffices to consider the case when the support of a is a compact interval $[c, d]$.

$$\begin{aligned}\left|\int ae^{i\phi}dx\right| =& \left|\int a/\phi' \frac{d}{dx}e^{i\phi}dx\right| \\ =& \left|\int e^{i\phi}\frac{d}{dx}(a/\phi')\right| \\ \le& \sup|a|\left|\frac{1}{\phi'(d)} - \frac{1}{\phi'(d)}\right| + \tau^{-1}\int |a'| \\ \le& \frac{3}{2}\tau^{-1}\int |a'|dx\end{aligned}$$

We use induction in k on the inequality

$$\left|\int a(x)e^{i\phi(x)}dx\right| \le 2k\tau^{-1/k}\left(\|a\|_{sup} + \|a'\|_{L^1}\right).$$

Suppose that the estimate holds for $k-1 \ge 1$ and we want to prove it for k. Suppose that there is point ξ_0 with $\phi^{(k-1)}(\xi_0) = 0$. We decompose the interval $[c, d]$ into $[c, \xi_0 - \delta]$, $[\xi_0 - \delta, \xi_0 + \delta]$ and $[\xi_0 + \delta, d]$. Then, by induction

$$|I| \le 2\delta\|a\|_{sup} + 2(k-1)(\delta\tau)^{-1/(k-1)}(\|a\|_{sup} + \|a'\|_{L^1}).$$

We choose $\delta = \tau^{-\frac{1}{k}}$. Then

$$|I| \le 2k\tau^{-\frac{1}{k}}\tau^{-\frac{1}{k}}\left(\|a\|_{sup} + \|a'\|_{L^1}\right)$$

which implies the desired inequality. □

1.4. **Examples and dispersive estimates.**

1.4.1. *The Schrödinger equation.* We consider the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

A Fourier transform (see next section) gives

$$i\partial_t \mathcal{F}_x u - |\xi|^2\mathcal{F}_x u = 0$$

and hence the unique solution in the space of tempered distributions is given by its Fourier transform

$$\mathcal{F}_x u(\xi) = e^{-it|\xi|^2}\mathcal{F}_x u.$$

Then

$$\frac{1}{(2\pi)^{d/2}}\int e^{-it|\xi|^2}d\xi = \frac{1}{\sqrt{2it}^d}.$$

Moreover a change of coordinates shows that

$$\frac{1}{(2\pi)^{d/2}}\int e^{-i(t|\xi|^2 - x\xi)}d\xi = e^{i\frac{x^2}{4t}}\int e^{it\xi^2}d\xi = \frac{1}{\sqrt{2it}^d}e^{i\frac{x^2}{4t}}. \tag{1.7}$$

Again we supress the approximation by a positive definite real part, and the corresponding limit.

1.4.2. *The Airy function and the Airy equation.* We consider the Airy equation

$$u_t + u_{xxx} = 0.$$

The Fourier transform transforms the equation to

$$\mathcal{F}_x u_t = (ik)^3 \mathcal{F}_x u$$

and hence, as above

$$\mathcal{F}_x u(t,\xi) = \mathcal{F}_x e^{it\xi^3} u(0)(\xi)$$

The Airy function is defined by

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int e^{i\frac{1}{3}\xi^3 + ix\xi} d\xi$$

where the right hand side has to be understood (as usual) as

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int e^{i\frac{1}{3}\xi^3 - \varepsilon|\xi|^2 + ix\xi} d\xi.$$

As above for the quadratic phase function we see that the limit exists at every point.

The phase function is

$$\phi(\xi) = \frac{1}{3}\xi^3 + x\xi$$

has the critical points

$$\xi^2 = -x.$$

If x is negative there are two real critical points.

We choose $a \in C^\infty(\mathbb{R})$, supported in $[-1,\infty)$ and identically 1 in $[1,\infty]$, with $\eta(\xi) + \eta(-\xi) = 1$. Then $\mathrm{Ai}(x)$ is the real part of

$$\frac{1}{2\pi} \int \eta e^{i(\frac{1}{3}\xi^3 + x\xi)} d\xi$$

There is no harm from the noncompact interval of integration and we to apply the stationary phase, Theorem 1.13 for $x \to -\infty$. The Hessian at the stationary points is $\tau := 6(-x)^{1/2}$ and we write

$$\phi(\xi) = \tau(\phi_0(\xi - (-x)^{1/2}))$$

where

$$\phi_0(\eta) = \left[\frac{6\tau}{e} ta^3 + \frac{1}{2}\eta^2\right]$$

which satisfies

$$\phi_0'(0) = 0, \phi_0''(0) = 1, \phi_0''' = [-6x]^{-1/2}.$$

After a shift we write the integral as

$$\frac{1}{2} e^{i(-\frac{2}{3})|x|^{\frac{3}{2}}} \int \eta(\eta + (-x)^{1/2}) e^{i\tau\phi_0(\eta)} d\eta$$

The application of the stationary phase theorem, 1.13, gives

$$\left| \mathrm{Ai}(x) - \frac{1}{\sqrt{\pi}} |x|^{-1/4} \cos(\frac{2}{3}|x|^{\frac{3}{2}} - \frac{\pi}{4}) \right| \le c|x|^{-\frac{7}{4}}$$

and there is even an asymptotic series. To see the error term we compute the next term, the sixth derivative of $e^{i\phi_0(\eta)}$, evaluated at 0. It gives an additional factor $\tau^{-3} = |x|^{-\frac{3}{2}}$.

For large positive x we need a different idea. For positive x there is fast decay and we want to determine the leading term. In this case the two critical points are purely imaginary real, and we shift the contour of integration to

$$\xi + i\sqrt{x}.$$

To be more precise we define

$$\mathrm{Ai}_\sigma(x) = \frac{1}{2\pi}\int e^{i[\frac{1}{3}(\xi+i\sigma)^3+x(\xi+i\sigma)]}$$

We expand

$$i[\frac{1}{3}(\xi+i\sigma)^3 + x(\xi+i\sigma) = i(\frac{1}{3}\xi^3 + x\xi - \xi\sigma^2) - \sigma(\xi^2 + x - \frac{1}{3}\sigma^2).$$

We calculate using the Cauchy Riemann equations

$$\frac{d}{d\sigma}\mathrm{Ai}_\tau(x) = \frac{1}{2\pi} = \int i\frac{\partial}{\partial\xi} e^{i(\xi^3+x\xi-\xi\sigma^2)-\sigma(\xi^2+x-\frac{1}{3}\sigma^2)}d\xi = 0$$

and hence

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int e^{i\frac{1}{3}\xi^3 - \sqrt{x}\xi^2 - \frac{2}{3}x^{\frac{3}{2}})}d\xi$$

with the critical point $\xi = 0$, at which point the Hessian is $\sqrt{x}$. We argue as above and obtain

$$\left|\mathrm{Ai}(x) - \frac{1}{2\sqrt{\pi}}|x|^{-1/4}e^{-\frac{3}{2}x^{\frac{3}{2}}}\right| \le c|x|^{-\frac{7}{4}}e^{-\frac{3}{2}x^{\frac{3}{2}}}. \tag{1.8}$$

The lemma of van der Corput ensures that the function Ai is bounded. More is true: About half a derivative of the Airy function is bounded in the following sense:

Lemma 1.15.

$$\left|\int |\xi|^{1/2}e^{i(\xi^3+x\xi)}d\xi\right| \le C$$

This is left as an exercise.

1.4.3. *Bessel functions. This was not part of the lecture. Nevertheless I want to keep the material here.*

The Bessel functions are confluent hypergeometric functions. They are solutions to confluent hypergeometric differential equations. Here is a very brief introduction.

Consider a complex differential equation

$$x^{(n)} = \sum_{j=0}^{n-1} a_j(z)z^{(j)}$$

with initial data

$$x^{(j)}(z_0) = y_j$$

for $j = 1 \dots n-1$ and given complex numbers z_0 and y_j. If the coefficients are holomophic in a neighborhood of z_0 then there is a unique solution which is holomorphic in z and the y_j.

Consider the scalar equation

$$\dot{x} = \frac{\lambda}{z - z_0}x$$

The space of solutions is at most 1 dimensional. Formally a solution is given by $x = (z - z_0)^\lambda$, which, unless z is an integer, is only defined in a set $\mathbb{C}\backslash(-\infty, z_0]$ called slit domain. Similarly, if

$$\dot{x} = (\frac{\lambda}{z - z_0} + \phi(z))x$$

with a holomorphic function ϕ near z_0 there is a unique solution of the type

$$(z - z_0)^\lambda \left[1 + \sum_{k=1}^\infty a_k (z - z_0)^k\right]$$

again defined in the slit domain as above unless λ is an integer. The number λ is called characteristic number. It is not hard to see that there is a unique such solution, and the power series can be iteratively defined. The point z_0 is called a regular singular point. A point is called irregular singular point if the Laurent series of the coefficients contains terms below $(z - z_0)^{-1}$

We call ∞ regular point resp. regular singular resp. irregular singular point for

$$\dot{x} = a(z)x$$

if, when we express z in terms of z^{-1}, 0 is a regular resp. regular singular or irregular singular point of

$$\dot{x} = -z^{-2}a(z^{-1})x$$

We use the same notation for systems of equations. The eigenvalues of A in

$$\dot{x} = \frac{1}{z - z_0} A(z)x + f(z)x$$

are called characteristic values. They play a very similar role as for scaler equations. Multiple characteristic values and or resonances (a resonance denotes the situation when eigen values of A are linearly dependend over the integers) may lead to logarithmic terms.

We are interested in second order scalar equations

$$a(z)\ddot{x} + b(z)\dot{x} + c(z)x = 0$$

with meromorphic functions a, b and c. We may rewrite them as a 2×2 system, which we use to define the notion of a regular, regular singular, and irregular singular term. The point z_0 is regular if $b(z)/a(z)$ and $c(z)/a(z)$ have a holomorphic extension near z_0. It is a regular singular point if the Laurent expansion of $b(z)/a(z)$ begins with $c_0 z^{-1}$ and the one of $c(z)/a(z)$ begins with $c_1 z^{-2} + c_2 z^{-1}$. The characteristic numbers can be calculated in terms of the Laurent series. If there are independent over the integers then there are unique solutions of the type

$$z^\gamma \sum a_j z^j$$

where γ is one of the characteristic numbers.

Of particular importance is the case when there are only regular singular points. In that case there are exactly three of them, and applying a Moebius transform we may choose them to be 0, 1 and ∞. Moreover, multiplying by $z^\lambda (1 - z)^\mu$ we can ensure that one of the characteristic values at 0 and 1 is 0. These are the hypergeometric differential equations

$$z(1 - z)\frac{d^2}{dz^2}w + [c - (a + b + 1)z]\frac{dw}{dz} - abw = 0$$

The characteristic numbers at $z = 0$ are 0 and $1 - c$, the ones at $z = 1$ are 0 and $c - a - b$, and the ones at infinity are $-a$ and $-b$.

The regular solution near 0 is denoted by

$$ {}_2F_1(a, b; c; z). $$

The Bessel differential equation is

$$ z^2\ddot{w} + w\dot{w} + (z^2 - \nu^2)w = 0. $$

It has a regular singularity at $z = 0$ with indices $\pm\nu$, and an irregular singularity at $z = \infty$. The Bessel function of the first kind is

$$ J_\nu = (\frac{1}{2}z)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu + k + 1)} $$

We have unless ν is negative integer

$$ J_\nu(z) - (\frac{1}{2}z)^\nu/\Gamma(\nu + 1) = O(|z|^{\operatorname{Re}\nu+1}) \text{ near } 0 $$

$$ J_\nu(z) - \sqrt{\frac{2}{\pi z}}\cos(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + e^{|\operatorname{Im} z|}o(1) $$

for $z \to \infty$ and $\nu \in \mathbb{R}$.

There is an integral representation for $\nu > -\frac{1}{2}$,

$$ \begin{aligned} J_\nu(z) =& \frac{2(\frac{1}{2}z)^\nu}{\pi^{1/2}\Gamma(\nu + \frac{1}{2})} \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \cos(zt)dt \\ =& \frac{(\frac{1}{2}z)^\nu}{\pi^{1/2}\Gamma(\nu + \frac{1}{2})} \int_0^\pi \cos(z\cos(\theta))\sin(\theta)^{2\nu}dt \end{aligned} $$

and Schläfli-Sommerfeld, if the absolute value of the argument of z is bounded by $\frac{1}{2}\pi$,

$$ J_\nu(z) = \frac{1}{2\pi i} \int_{-\infty-\pi i}^{\infty+\pi i} e^{z\sinh t - \nu t}dt $$

$$ \begin{aligned} J_\nu(z) =& \frac{2(\frac{1}{2}z)^\nu}{\pi^{1/2}\Gamma(\nu + \frac{1}{2})} \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \cos(zt)dt \\ =& \frac{(\frac{1}{2}z)^\nu}{2\pi i} \int_{-\infty}^{0+} \exp(t - \frac{z^2}{4t})t^{-\nu+1}dt \end{aligned} $$

They satisfy

$$ \left(\frac{d}{xdx}\right)^m (x^\nu J_\nu) = x^{\nu-m}J_{\nu-m}. $$

See [24] for more information. We want to evaluate (with the Hausdorff measure of dimension s denoted by $\mathcal{H}^s$

$$ \begin{aligned} \int_{\mathbb{S}^{d-1}} e^{ix\xi}d\mathcal{H}^{d-1} =& \int_0^\pi \mathcal{H}^{d-2}(\mathbb{S}^{d-2})\sin^{d-2}(\theta)e^{i|x|\cos(\theta)}d\theta \\ =& J_{\frac{d-2}{2}}(|x|)\pi^{\frac{d-1}{2}}(\frac{1}{2}|x|)^{-\frac{d-2}{2}} \end{aligned} $$

This function is real and radial and we define - using stereographic coordinates

$$ \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} = \begin{pmatrix} \frac{2y_1}{1+|y|^2} & \frac{2y_2}{1+|y|^2} & \dots & \frac{-1+|y|^2}{1+|y|^2} \end{pmatrix} $$

with Jacobian determinant

$$\frac{2^{d-1}}{(1+|y|^2)^{d-1}}$$

We choose a suitable smooth function η supported in $[-1/2,\infty)$, identically 1 in $[\frac12,\infty)$ with $\eta(-t)+\eta(t)=1$, and apply the area formula

$$\begin{aligned}\int e^{ix\cdot\xi} d\mathcal{H}^{d-1}(\xi) &= \mathrm{Re} \int \eta(-\xi_n) e^{i|x|\xi_n} d\mathcal{H}^{d-1} \\ &= \mathrm{Re} \int_{\mathbb{R}^{d-1}} \eta(\frac{-1+|y|^2}{1+|y|^2}) e^{i|x|\frac{-1+|y|^2}{1+|y|^2}} \frac{2^{d-1}}{(1+|y|^2)^{d-1}} dm^{d-1}(y)\end{aligned}$$

and an application of stationary phase gives

Lemma 1.16. *For all $k\in\mathbb{N}$*

$$\left| \left(\frac{d}{dr}\right)^k \left\{ e^{-ir} \int_{\mathbb{R}^{d-1}} \eta(\frac{1-|y|^2}{1+|y|^2}) e^{ir\frac{-1+|y|^2}{1+|y|^2}} \frac{4}{(1+|y|^2)^2} dm^{d-1}(y) \right\} \right| \le c_k r^{-\frac{d-1}{2}-k}.$$

Proof: Exercise.

1.5. **The Fourier transform.** Let f be an integrable complex valued function. We define its Fourier transform by

$$\hat f(\xi) = \frac{1}{(2\pi)^{d/2}} \int e^{-ix\cdot\xi} f(x) dm^d(x). \tag{1.9}$$

1.5.1. *The Fourier transform in L^1.* Properties are

1) The Fourier transform of an integrable function is a bounded continuous function which converges to 0 as $|\xi|\to\infty$. It satisfies

$$\|\hat f\|_{sup} \le (2\pi)^{-d/2}\|f\|_{L^1}.$$

The estimate is obvious, as is the continuity if f is compactly support. The limit as $x\to\infty$ follows by an integration by parts if the integrand is compactly supported and differentiable. Those functions are dense, and we obtain continuity and vanishing of the limit for compactly supported functions. The limit

$$\lim_{R\to\infty} \int_{B_R(0)} e^{-ix\cdot\xi} f(x) dm^d(x)$$

is uniform, and hence the Fourier transform is continuous and converges to 0 as $\xi\to\infty$.

2) For all η and y in $\mathbb{R}^d$

$$\hat f(\xi+\eta) = \widehat{e^{-i\eta\cdot x} f} \tag{1.10}$$

and

$$\widehat{f(.+y)} = e^{iy\xi}\hat f(\xi). \tag{1.11}$$

This follows by an simple calculation.

3) For $f,g\in L^1(\mathbb{R})$

$$\widehat{f*g}(\xi) = (2\pi)^{n/2}\hat f(\xi)\hat g(\xi).$$

which follows by application of Fubini's theorem:

$$\begin{aligned}\frac{1}{(2\pi)^{d/2}}\int e^{-ix\xi}\int f(y)g(x-y)dm^d(y)dm^d(\xi)&\\ &=\int\int e^{-iy\xi}f(y)e^{-i(x-y)\xi}g(x-y)dm^d(y)dm^d(x)\\ &=\int\int e^{-iy\xi}f(y)e^{-iz\xi}g(z)dm^d(z)dm^d(y)\\ &=(2\pi)^{d/2}\hat f(\xi)\hat g(\xi)\end{aligned}$$

4) For f and $g \in L^1$

$$\int f\hat g dm^d(x) = \int \hat f g dm^d \tag{1.12}$$

This is seen by applying Fubini to

$$\int\int e^{-iy\xi}f(y)e^{-i(x-y)\xi}g(y)dm^d(y)dm^d(x).$$

5)

$$\widehat{e^{-\frac12|x|^2}} = e^{-\frac12|\xi|^2}$$

We calculate as above

$$(2\pi)^{-d/2}\int e^{-ix\xi-\frac12|x|^2}dm^d(x) = (2\pi)^{-d/2}\int e^{-i(x-i\eta)\xi-\frac12(x-i\eta)^2}dm^d(x)$$

for $\eta\in\mathbb{R}^n$. We set $\eta=\xi$ and get

$$(2\pi)^{-d/2}e^{-\frac{|\xi|^2}{2}}\int e^{-\frac12|x|^2}dx = e^{-\frac{|\xi|^2}{2}}.$$

1.5.2. *The Fourier transform of Schwartz functions.*

Definition 1.17. *We say $f\in C^\infty(\mathbb{R}^d)$ is a Schwartz function and write $f\in\mathcal{S}(\mathbb{R}^d)$ if for all multiindices α and β*

$$\|x^\alpha\partial^\beta f\|_{sup}<\infty$$

We say $f_j\to f$ in $\mathcal{S}$ if for all multiinidices

$$x^\alpha\partial^\beta f_j\to x^\alpha\partial^\beta f$$

uniformly.

We collect elementary properties.
1) $f\in\mathcal{S}$ if and only if $x^\alpha\partial^\beta f\in\mathcal{S}$ for all α and β.
2) $f\in\mathcal{S}$ implies f integrable.
3) $f\in\mathcal{S}$ and $g\in C^\infty$ with bounded derivatives implies $fg\in\mathcal{S}$.
4) $f\in\mathcal{S}$ and A an invertible $d\times d$ matrix implies $f\circ A\in\mathcal{S}$
5) $f\in\mathcal{S}$ and $x_0\in\mathbb{R}^d$ implies $f(.+x_0)\in\mathcal{S}$.
6) We say that a distribution T has compact support, if there exists a ball $B_R(0)$ such that for all functions f in $C_0^\infty(\mathbb{R}^d)$ with support disjoint from $B_R(0)$ $Tf=0$. We can easily extend such distributions to Schwartz functions (exercise).

We define the convolution with a Schwartz function by

$$T*f(x)=T(f(x-.))$$

This is well defined and $T * f$ is a Schwartz function whenever f is a Schwartz function. To see this we recall that by the definition of a distribution there exist $C > 0$ and $N > 0$ such that (since f has compact support)

$$|T(f)| \le c_N \|f\|_{C^N}.$$

Taking difference quotients shows that $x \to T * f(x)$ is differentiable and

$$\partial_i T * f = T * \partial_i f.$$

Recursively we see that $Tf \in C^\infty$. Morever

$$\|f(x - .)\|_{C^N(B_R(0))} \le c_M (1 + |x|)^{-M}$$

for Schwartz functions, and hence $T * f$ is a Schwartz function.

7) $f, g \in \mathcal{S}$ implies $f * g \in \mathcal{S}$ and

$$\widehat{f * g} = (2\pi)^{d/2} \hat{f} \hat{g} \tag{1.13}$$

If $f \in \mathcal{S}$ and S is a distribution with compact support then

$$S * f(x) := S(f(x - .)) \in \mathcal{S}.$$

8) All the operations above are continuous.

Theorem 1.18. *If $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$, and vice verse,*

$$\widehat{x_j f} = -i \partial_{\xi_j} \hat{f}$$

$$\widehat{-i\partial_{x_j}} f = \xi_j \hat{f}$$

and the Fourier inversion formula

$$f(x) = (2\pi)^{-d/2} \int e^{ix\xi} \hat{f}(\xi) dm^d(\xi)$$

and the Plancherel formula

$$\int \hat{f}\overline{\hat{g}} dm^d(\xi) = \int f \overline{g} dm^d(x)$$

hold. If A is a real invertible $d \times d$ matrix then

$$\widehat{f \circ A}(\xi) = (\det |A|)^{-1} \hat{f}(A^{-T}\xi).$$

Proof. According to property (1)

$$x^\alpha \partial^\beta f \in \mathcal{S}$$

and hence $x^\alpha \partial^\beta f$ is integrable. With the first calculation

$$x^\alpha \widehat{(-i\beta^\beta} f) = -i \partial^\alpha \xi^\beta \hat{f}$$

which is bounded by the second observation. Thus $\hat{f} \in \mathcal{S}$. We calculate

$$\mathcal{F}((2\pi)^{-d/2} \tau^{d/2} e^{-\frac{\tau}{2}x^2} * f) = e^{-\frac{1}{2\tau}\xi^2} \hat{f}(\xi)$$

and, with $\tau \to \infty$

$$f(0) = (2\pi)^{-d/2} \int \hat{f} d\xi.$$

Together with the formulas (1.11) we obtain the inversion formula

$$f(x) = (2\pi)^{-d/2} \int e^{ix\xi} \hat{f}(\xi) d\xi.$$

The Plancherel formula follows by (1.12). The last formula follows from

$$(2\pi)^{-d/2}\int e^{-ix\cdot\xi} f(Ax)dm^d(x) = (2\pi)^{-d/2}|\det A|^{-1}\int e^{-i(A^{-1}y)\cdot\xi} f(y)dm^d(y).$$

□

1.5.3. *Tempered distributions.*

Definition 1.19. *A tempered distribution T is a linear map*

$$T : \mathcal{S} \to \mathbb{C}$$

which is continuous, i.e. $f_j \to f \in \mathcal{S}$ implies

$$Tf_j \to Tf$$

We denote the set of tempered distributions as $\mathcal{S}^*$. We say T_j converges to T if $T_j f \to Tf$ for all $f \in \mathcal{S}$.

We list properties.

1) We call T bounded if there exists N such that

$$|Tf| \le C \sup_{|\alpha|+|\beta|\le N} \sup_x |x^\alpha \partial_x^\beta f|.$$

The linear $T : \mathcal{S} \to \mathbb{C}$ is bounded if and only if it is continuous.

2) Distributions with compact support are tempered distributions.

3) Let $T \in \mathcal{S}^*$ and $\phi \in C^\infty$ with bounded derivatives. We define

$$\phi T(f) = T(\phi f).$$

4) The derivative of a tempered distribution $\partial_j T$ is defined by

$$\partial_j T(f) = -T(\partial_j f)$$

5) Let $T \in \mathcal{S}^*$ and $\phi \in \mathcal{S}$. Then

$$T * \phi \in C^\infty(\mathbb{R}^d),$$

where we define $T * \phi$ as for distributions with compact support.

6) Let $T \in \mathcal{S}^*$ and S be a distribution with compact support. We define

$$S * T(f) = T(\tilde{S} * f)$$

where $\tilde{S}(f) = S(\tilde{f})$, $\tilde{f}(x) = f(-x)$. Then $S * T \in \mathcal{S}^*$.

7) Let $g \in L^p$ for one $1 \le p \le \infty$. It defines a unique distribution by

$$T_g(f) = \int g f dm^d.$$

The operations commute with this representation,

$$T_{\phi g} = \phi T_g$$

and we identify L^p with its image via the embedding.

8) We define the Fourier transform $\hat{T} \in \mathcal{S}^*$ by

$$\hat{T}(f) = T(\hat{f})$$

The inverse Fourier transform is defined similarly.

This is compatible with the interpretation for functions.

9)

$$\hat{\delta}_0 = (2\pi)^{d/2}$$

and

$$\hat{1} = (2\pi)^{d/2}\delta_0$$

Lemma 1.20. *The following formula holds for all integrable radial functions which we write by an abuse of notation $f(|x|)$:*

$$(2\pi)^{-d/2}\int e^{-ix\cdot\xi} f(|x|)dm^d(x) = (2\pi)^{-1/2}|\xi|^{1-\frac{d}{2}}\int_0^\infty r^{d/2}f(r)J_{\frac{d-2}{2}}(r|\xi|)dr$$

In three dimensions the formula becomes particularly simple since

$$J_{\frac12}(z) = \sqrt{\frac{2}{\pi}}\frac{\sin z}{z^{\frac12}}$$

Thus

$$(2\pi)^{-\frac32}\int f(|x|)e^{-ix\xi}dx = \sqrt{\frac{2}{\pi}}|\xi|^{-1}\int_0^\infty f(r)r\sin(r|\xi|)dr.$$

The Euler relation

$$x\cdot\nabla f = mf$$

holds for every homogeneous function of degree m. We want to define homogeneous distributions.

Definition 1.21. *A tempered distribution is called homogeneous of degree $m \in \mathbb{C}$ if*

$$T(\phi) = \lambda^{-d-m}T(\phi(\lambda * .)).$$

Let $\operatorname{Re} m > -d$. Then $|x|^m$ is tempered distribution. Its Fourier transform is again a tempered distribution of homogeneity $-d-m$.

This can be seen from the Euler relation

$$x\cdot\nabla f = mf$$

for every homogeneous function of degree m.

Lemma 1.22. *Let $0 < \operatorname{Re} m < d$. The following identity holds*

$$\mathcal{F}(\frac{1}{2^{m/2}\Gamma(m/2)}|x|^{m-d}) = \frac{1}{2^{(d-m)/2}\Gamma(\frac{d-m}{2})}|x|^{-m}.$$

Proof. We claim that the Fourier transform of a homogeneous distribution of degree $m\in\mathbb{C}$ is a homogeneous distribution of degree $-d-m$. We denote by T_λ the distribution

$$T_\lambda(f) = \lambda^{-d}Tf(\lambda.)$$

Then

$$\hat{T}_\lambda(f) = T_\lambda(\hat{f}) = T(\lambda^{-d}\hat{f}(\lambda.)) = T(\widehat{f(./\lambda)}) = \lambda^{-m-d}T(\hat{f}) = \lambda^{-m-d}\hat{T}(f).$$

Let f be a homogeneous function of degree m such that T_f is a homogeneous distribution. Let O be an orthogonal matrix with $f\circ O = f$. Then

$$\widehat{T}_f\circ O^T = \widehat{T}_f$$

where the term on the left hand side is defined by the action on Schwartz functions. In particular the Fourier transform of $|x|^{-m}$ is radial in the sense that it is invariant under the action of orthogonal matrices. This is equivalent to

$$Tf = T\left(\mathcal{H}^{d-1}(\mathbb{S}^{d-1})^{-1}\int_{\mathbb{S}^{d-1}} f(|x|\sigma)\,\mathcal{H}^{d-1}(\sigma)\right)$$

(a rigorous justification requires either an approximation, or a symmetrization argument). We denote the symmetrization operator by S.

Let T be a radial homogeneous distribution of degree m. We claim that We fix a nonnegative function h with integral 1 with compact support and observe that

$$\begin{aligned} T(f) =& T\Big(\int_0^\infty \lambda^{-d-m-1}(Sf)(\lambda x)h(\ln(\lambda))d\lambda\Big) \\ =& T(\int_0^\infty \lambda^{-d-m-1}Sf(\lambda)|x|^{d+m}h(\ln(\lambda/|x|))d\lambda \\ =& T(|x|^{d+m}h(-\ln|x|)\int_0^\infty \lambda^{-d-m-1}Sf(\lambda)d\lambda \\ =& T\left(|x|^{d+m}h(-\ln|x|)\right)\int |y|^m f(y)dm^d(y) \end{aligned}$$

for all $f \in \mathcal{S}$ with 0 not in the support. This extends to Schwartz functions if $m > -d$.

By the consideration above

$$\widehat{|x|^{-m}} = c(n,m)|x|^{m-d}$$

and we have to determine $c(n,m)$. The Gaussian is its own Fourier transform. Let $T = |x|^m$ and denote by $\hat{T}$ its Fourier transform. Then, by the definition

$$T(e^{-\frac{|x|^2}{2}}) = \hat{T}(e^{-\frac{|\xi|^2}{2}})$$

We calculate

$$\begin{aligned} \int |x|^m e^{-\frac{|x|^2}{2}} dm^d(x) =& d\, m^d(B_1(0))\int_0^\infty e^{-r^2/2}r^{d-1+m}dr \\ =& d\, m^d(B_1(0))2^{-\frac{d+m}{2}-1}\int_0^\infty t^{\frac{d+m}{2}-1}e^{-t}dt \\ =& d\, m^d(B_1(0))2^{-\frac{d+m}{2}-1}\Gamma(\frac{d+m}{2}). \end{aligned}$$

Comparison with the calculation for $|x|^{-d-m}$ gives the formula. □

The formula extends to all $m \in \mathbb{C}\backslash(-\infty,-d] \cup [0,\infty)$. This requires however a proper definition of the homogeneous tempered distribution.

1.6. **Examples of Fourier transforms.**

1.6.1. *Laplacian and related operators.* Let $d > 2$. Then

$$\widehat{|x|^{2-d}} = \frac{1}{2^{(d-4)/2}\Gamma(\frac{d-2}{2})}|\xi|^{-2}$$

for some explicit c_d and

$$-\Delta(4\pi)^{d/2}\frac{1}{\Gamma(\frac{d-2}{2})}\int |x-y|^{2-d}f(y)dy = f(y).$$

The Fourier transform transforms derivatives into multiplication by polynomial functions. For example

$$\widehat{f-\Delta f} = (1+|\xi|^2)\hat{f}$$

and hence

$$\hat{u} = (1+|\xi|^2)^{-1}\hat{f}$$

is the Fourier transform of a Schwartz function u (if f is a Schwartz function) which satisfies

$$-\Delta u + u = f.$$

Here $(1+|\xi|^2)^{-1}$ is a smooth function with bounded derivatives, but not a Schwartz function. Its inverse Fourier transform k allows to define a solution for a given function f by

$$u = (2\pi)^{d/2} k * f$$

We compute k for several space dimensions.

$$\int_{-\infty}^{\infty} e^{ix\xi}(1+\xi^2)^{-1} d\xi = \pi e^{-|x|} \tag{1.14}$$

using the residue theorem: The singular points are the zeroes of the polynomial $1+\xi^2$, which are $\pm i$. Consider the case $x>0$ first. By the residue theorem

$$\int_{C_R} e^{ix\xi}(1+\xi^2)^{-1} d\vec{\xi} = \pi e^{-|x|}$$

where C_R is the union of the path from $-R$ to R and the upper semi circle. The limit $R \to \infty$ implies the statement.

In three dimensions, by Lemma 1.20 the Fourier transform is given by

$$\sqrt{\frac{2}{\pi}} |\xi|^{-1} \int_0^\infty (1+r^2)^{-1} \sin(r|\xi|) dr = \pi |x|^{-1} e^{-|x|}.$$

1.6.2. *Gaussians, heat and Schrödinger equation.*

Lemma 1.23. *Let $A = A_0 + iA_1$ be an invertible symmetric matrix (A_0 and A_1 real) with A_0 positiv semidefinit. Then*

$$\mathcal{F} e^{-\frac{1}{2}x^T A x}(\xi) = \det(A)^{-1/2} e^{-\frac{1}{2}\xi^T A^{-1}\xi}.$$

Proof. The formula is correct at $\xi = 0$. We assume first that A_0 is positive definite. The general statement follows then by continuity of both sides. By definition

$$\nabla e^{-\frac{1}{2}x^T A x} + e^{-\frac{1}{2}x^T A x} A x = 0$$

The Fourier transform g is a Schwarz function which then satisfies

$$g\xi + A\nabla g = 0.$$

This is an ordinary differential equation on lines through the origin. There is a unique solution with the given value at $\xi = 0$, which has to coincide with the function on the right hand side. □

With $A = 2t 1_{\mathbb{R}^d}$ we obtain the formula for the fundamental solution to the heat equation. The inverse Fourier transform of $e^{-it|\xi|^2}$ is - as computed twice -

$$(2it)^{d/2} e^{-\frac{|x|^2}{4it}}$$

A solution to the Schrödinger equation

$$iu_t + \Delta u = 0$$

with initial data u_0 is given by

$$u(t,x) = \int_{\mathbb{R}^d} (4i\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4it}} u_0(y) dy \tag{1.15}$$

We denote the map $u(0,.) \to u(t,.)$ by $S(t)$. It is defined by the Fourier transform by

$$\widehat{S(t)u_0} = e^{-it|\xi|^2}\hat{u}_0(\xi).$$

It is a unitary operator:

$$\|S(t)u_0\|_{L^2} = \|\widehat{S(t)u_0}\|_{L^2} = \|e^{-it|\xi|^2}\hat{u}_0\|_{L^2} = \|\hat{u}_0\|_{L^2} = \|u_0\|_{L^2}$$

and it satisfies the socalled dispersive estimate

$$\|S(t)u_0\|_{sup} \le |4\pi t|^{-d/2}\|u_0\|_{L^1}.$$

1.6.3. *The Airy equation.* The Airy function is the inverse Fourier transform of

$$\widehat{\mathrm{Ai}}(\xi) = (2\pi)^{-1/2}e^{i\frac{1}{3}\xi^3}$$

Clearly

$$(\xi^2 + i\partial_\xi)e^{i\frac{1}{3}\xi^3} = 0$$

and hence

$$\mathrm{Ai}'' + x\,\mathrm{Ai} = 0$$

This however implies

$$(\partial_t + \partial^3_{xxx})((t/3)^{-1/3}\,\mathrm{Ai}(x(t/3)^{-1/3})) = 0$$

and (as oscillatory integral)

$$\int \mathrm{Ai}(x)dx = (2\pi)^{-1/2}$$

The convolution be the Airy function gives a solution to the initial value problem

$$u_t + u_{xxx} = 0, \qquad u(0,x) = u_0(x),$$

$$u(t,x) = \int (t/3)^{-1/3}\,\mathrm{Ai}((x-y)(t/3)^{-1/3})u_0(y)dy.$$

Again the equation defines unitary operators $S(t)$ which satisfy

$$\|S(t)u_0\|_{sup} \le ct^{-1/3}\|u_0\|_{L^1}$$

1.6.4. *The half-wave equation.* The solution to the wave equation

$$u_{tt} - \Delta u = 0$$

with initial data

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x)$$

is given by Kirchhoff's formula for $d = 3$:

$$u(t,x) = \frac{1}{4\pi t^2}\int_{\partial B_t(x)} u_0 d\mathcal{H}^2 + \frac{1}{4\pi t}\int_{\partial B_t(x)} \partial_\nu u_0 d\mathcal{H}^2 + \frac{1}{4\pi t}\int_{\partial B_t(x)} u_1 d\mathcal{H}^2.$$

There are similar formulas in odd dimensions, and slightly more complicated ones in even dimensions.

The Fourier transform tranforms the PDE to the ODE

$$\hat{u}_{tt} + |\xi|^2\hat{u} = 0$$

which factorizes into

$$(\partial_t - i|\xi|)(\partial_t + i|\xi|) = 0.$$

This motivated the study of the half wave equation

$$(i\partial_t + |\xi|)\hat{u}(t,\xi) = 0$$

which can easily be solved in the form

$$\hat{u}(t,\xi) = e^{it|\xi|}\hat{u}(0,\xi).$$

As above we restrict to $t=1$. Since $e^{it|\xi|}$ is radial

$$\int e^{i(|\xi|+x\xi)}d\xi = dm^d(B_1(0))|x|^{-\frac{d-2}{2}}\int_0^\infty r^{d/2}e^{ir}J_{\frac{d-2}{2}}(|x|r)dr$$

provided the integrals exist as oscillatory integrals. They do as we will see. By Lemma 1.16 we can write

$$z^{\frac{d-1}{2}}J(z) = \operatorname{Re}(e^{-iz}\phi(z))$$

for $z \geq 1$, with ϕ satisfying

$$|\phi^{(k)}(z)| \leq c_k z^{-k}.$$

We begin to consider $|x| \geq 2$ We decompose the integral above into two parts with a smooth cutoff function, one over $r \geq |x|^{-1}$, and one over $2|x|^{-1}$. In the first integral we integrate by parts as often as we like:

$$\int_0^\infty (1-\eta(r|x|))e^{ir(1\pm|x|)}p(rx)dx = \frac{i}{1\pm x}\int_0^\infty e^{ir(1\pm|x|)}(\frac{d}{dr}((1-\eta(r|x|))p(rx))dx$$

which gains a factor r in the integration, as well as a power $|x|^{-1}$. We repeat this as often as necessary. The second integral is bounded by $|x|^d$.

The same arguments apply as for $|x| \neq 1$, given bounds which depend only on $|x|-1$. A careful calculation gives the first part of the following estimate

Lemma 1.24. *Suppose that $|x| \neq 1$ then*

$$\left|\int e^{i|\xi|+ix\xi}d\xi\right| \leq \begin{cases} c_d|1-|x||^{-\frac{d+1}{2}} & \text{if } |x| \leq 2 \\ c_d|x|^{-d} & \text{if } |x| \geq 2 \text{ and } d \text{ even} \end{cases}$$

and

$$\left|\int |\xi|^{-\frac{d+1}{2}}e^{i|\xi|+ix\xi}d\xi - c\ln|1-|x||\right| \leq c_d$$

if $|x| \leq 2$ and

$$\left|\int |\xi|^{-\frac{d+1}{2}}e^{i|\xi|+ix\xi}d\xi\right| \leq c_d|x|^{-\frac{d-1}{2}}.$$

for $|x| \geq 2$.

Proof. Only the second part remains to be shown. There is no difference in for $|x| \leq 2$, unless $|x|$ is close to 1. In that case we decompose the integral into $r \leq 2$, $1 \leq r \leq |x|-1$ and $r \geq |x|-1$. The last part is bounded by the previous arguments. The first part is bounded because of the size $r \leq 1$. The second part is

$$\int_1^{||x|-1|} r^{-1}dr = \ln r$$

plus something bounded. □

There is an important difference compared to the previous two examples: the group velocity depends only on the direction of ξ, not on the amplitude.

1.6.5. *The Klein-Gordon half wave.* Let

$$g(t,x) = \int e^{it\sqrt{1+|\xi|^2}+ix\xi} d\xi.$$

As above we obtain

Lemma 1.25. *The following estimates hold for $t \geq 1$,*

$$|g(t,x)| \leq c \begin{cases} t^{-d/2}(1-|x|/t)^{-\frac{d+1}{2}} & \text{if } |x| < t \\ t^{-d}(|x|/t-1)^{-\frac{d+1}{2}} & \text{if } t < |x| \leq 2t \\ \frac{1}{|x|^d t^{d-1}} & \text{if } |x| \geq 2t \end{cases}$$

and if $0 < t < 1$

$$|g(t,x)| \leq c \begin{cases} t^{-d} & \text{if } |x| < t \\ t^{-d}(|x|/t-1)^{-\frac{d+1}{2}} & \text{if } t < |x| \leq 2t \\ \frac{1}{|x|^d t^{d-1}} & \text{if } |x| \geq 2t \end{cases}$$

Moreover

$$h = \int |\xi|^{-\frac{d+1}{2}} e^{it\sqrt{1+|\xi|^2}+ix\xi} d\xi$$

satisfies for $t \geq 1$ and $|x| \geq 2t$

$$|h(t,x)| \leq C \frac{1}{|x|^{\frac{d-1}{2}} t^{\frac{d}{2}-3}}$$

$$\Big| h(t,x) - ct^{\frac{1}{2}} |\ln||1-|x|/t|| \Big| \leq ct^{\frac{1}{2}}$$

for $1 \leq t$, $|x| \leq 2t$. Finally, if $0 < t \leq 1$, then

$$\left| \int |\xi|^{-\frac{d+1}{2}} e^{it|\xi|+ix\xi} d\xi - ct^{-\frac{d-1}{2}} \ln|1-|x|| \right| \leq c_d t^{-\frac{d-1}{2}}$$

if $|x| \leq 2t$ and

$$\left| \int |\xi|^{-\frac{d+1}{2}} e^{i|\xi|+ix\xi} d\xi \right| \leq c_d \frac{1}{|x|^{\frac{d-1}{2}} t^{\frac{d}{2}-3}}.$$

for $|x| \geq 2$.

1.6.6. *The Kadomtsev-Petviashvili equation.* The linear parts of the Kadomtsev-Petviashvili equations are

$$u_t + u_{xxx} \pm \partial_x^{-1} u_{yy} = 0.$$

We denote the Fourier variables by ξ and η. As above (for $+$, the argument for $-$ is very similar),

$$\mathcal{F}_{x,y} u(t,\xi,\eta) = e^{it(\xi^3 - \xi^{-1}\eta^2)} \mathcal{F}_{x,y} u(0,\xi,\eta)$$

and

$$\int e^{i[(\xi^3-\xi^{-1}\eta^2)+x\xi+y\eta]} d\xi d\eta = \int e^{i[\xi^3+\xi x+\xi y^2/4]} d\xi.$$

The Lemma of van der Corput with $k = 3$ ensures that the integral is bounded, uniformly in x and y. More precisely

$$\left| \int e^{i[\xi^3+\xi x+\xi y^2/4]} e^{-\varepsilon|\xi|^2} d\xi \right| \leq c \|(e^{-\varepsilon|\xi|^2})'\|_{L^1} = 2C$$

The stationary points of the phase function satisfy

$$3\xi^2 + x + y^2/4 = 0$$

with zeroes

$$\xi = \pm\sqrt{-(x+y^2/4)/3}$$

provided

$$x < -\frac{1}{4}y^2.$$

Otherwise, by the nondegeneracy of the phase

$$\left| \int e^{i[(\xi^3-\xi^{-1}\eta^2)+x\xi+y\eta]} d\xi d\eta \right| \le c_k |x+y^2/4|^{-k}$$

Lemma 1.26.

$$\left| \int e^{i\xi^3 \mp \eta^2/\xi + ix\xi} d\xi d\eta \right| \le c_k (1+(x\pm y^2)_+)^{-k}.$$

There is an interesting interpretation:

- Waves move to left for Kadomtsev-Petviashvili II,
- and to both sides for Kadomtsev-Petviashvili I (with respect to x)

This make the study of Kadomtsev-Petviashvili I considerably harder than the study of Kadomtsev-Petviashvili II.

We define

$$\rho(x,y) = 2\pi \mathcal{F}^{-1}(e^{i(\xi^3-\eta^2/\xi)}).$$

Since $u(\lambda^3 t, \lambda x, \lambda^2 y)$ satisfies the linear KP equation for $\lambda > 0$ if and only if u does we obtain the representation

$$u(t,x,y) = g_t * u(0,.,.)(x,y)$$

where

$$g_t(x,y) = t^{-1}\rho(x/t^{1/3}, y/t^{2/3}).$$

Hence, with $S(t)$ denoting the evolution operator,

$$\|S(t)u_0\|_{L^2} = \|u_0\|_{L^2}$$

and

$$\|S(t)u_0\|_{sup} \le c|t|^{-1}\|u_0\|_{L^1(\mathbb{R}^2)}.$$

2. Strichartz estimates and small data for the Nonlinear Schrödinger equation

2.1. Strichartz estimates for the Schrödinger equation. We return to the linear Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

and the unitary operators $S(t)$. They form a group: For $s, t \in \mathbb{R}$

$$S(t+s) = S(t)S(s).$$

We claim that for $2 \le p \le \infty$

$$\|S(t)\|_{L^p} \le (4\pi|t|)^{-\frac{d}{2}(1-\frac{2}{p})} \|u_0\|_{L^{p'}}, \tag{2.1}$$

which follows by complex interpolation from

$$\|S(t)u_0\|_{L^2} = \|u_0\|_{L^2}$$

and the dispersive estimate

$$\|S(t)u_0\|_{L^\infty} \le (4\pi|t|)^{-\frac{d}{2}} \|u_0\|_{L^1}$$

Let us be more precise. We put $p_0 = q_0 = 2$ and $p_1 = 1$, $q_1 = \infty$, $2 < \tilde{p} < \infty$ and determine λ so that

$$\frac{1-\lambda}{2} = \frac{1}{p},$$

resp.

$$\lambda = 1 - \frac{2}{p}$$

define

$$\frac{1-\lambda}{2} + \lambda = \frac{1}{q}.$$

We check easily

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and obtain by complex interpolation

$$\|S(t)u_0\|_{L^\infty} \le (4\pi|t|)^{-\frac{\lambda d}{2}} \|u_0\|_{L}.$$

which is estimate (2.1).

The variation of constants formula

$$u(t) = -i \int_{-\infty}^t S(t-s) f(s) ds$$

defines a solution to

$$i\partial_t u + \Delta u = f$$

at least for Schwartz functions f in $d+1$ variables.

From the dispersive estimate

$$\|u(t)\|_{L^{p'}} \le (4\pi)^{-\frac{d}{2}(1-\frac{2}{p})} \int_{-\infty}^t |t-s|^{\frac{d}{2}-\frac{d}{p}} \|f(s)\|_{L^p} ds.$$

The right hand side is a convolution $h * g$ where

$$h(t) = \begin{cases} 0 & \text{if } t > 0 \\ |4\pi t|^{-\frac{d}{2}(1-\frac{2}{p})} & \text{if } t < 0 \end{cases}$$

and

$$g(t) = \|f(t)\|_{L^{p'}(\mathbb{R}^d)}.$$

An immediate calculation gives $|t|^{-1/r} \in L^r_w(\mathbb{R})$ and by the weak Young inequality Proposition 1.2

$$\|g * h\|_{L^q(\mathbb{R})} \le c\|g\|_{L^{q'}}\|h\|_{L^r_w} \tag{2.2}$$

where

$$r\frac{d}{2}(1-\frac{2}{p}) = 1, r > 1.$$

and p and q are strict Strichartz pairs, i.e. numbers which satisfy

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2}.$$

and $2 < q \le \infty, 2 \le p \le \infty$. The left hand side of (2.2) controls

$$\|u\|_{L^q_t L^p_x} := \Big(\int \|u(t)\|^q_{L^p(\mathbb{R}^d)} dt\Big)^{1/q}$$

wich the obvious modification if $q = \infty$ and we obtain

$$\|u\|_{L^q_t L^p_x} \le c\|f\|_{L^{q'}_t L^{p'}_x}$$

for all strict Strichartz pairs. Here $L^q_t L^p_x$ consists of all equivalence classes of measurable functions such that the expression is finite.

It is not hard to see that u measurable implies

$$t \to \|u(t,.)\|_{L^p}$$

is measurable, the expression for the norm actually defines a norm, and the space is closed and hence a Banach space. The duality of the Lebesgue spaces extends to duality of this mixed norm spaces: The map

$$L^{p',q'} \ni f \to (g \to \int fg dm^d dt) \in (L^{p,q})^*$$

is an isometry if $1 \le p, q \le \infty$ and surjective if $p, q < \infty$. Complex interpolation extends to the mixed norm spaces - this is quite evident from the definition.

We claim

Theorem 2.1. *The variation of constants formula defines a function u which satisfies*

$$i\partial_t u + \Delta u = f, \qquad u(0) = u_0$$

and let (q,p) be a strict Strichartz pair. Then

$$\|u\|_{C_b(\mathbb{R},L^2)} + \|u\|_{L^q L^p} \le c\|u(0)\|_{L^2} + \|f\|_{L^{q'}L^{p'}}.$$

We will later improve this estimate in several directions.

Now denote

$$L^2 \ni v \to Tv \in C([0,\infty), L^2)$$

which maps the initial data to the solution. Let (p,q) be Strichartz pairs. Then

$$\|T\|^2_{L(L^2,L^{q'}L^{p'})} = \|T^*\|^2_{L(L^{q,p},L^2)} = \|TT^*\|_{L(L^{q,p},L^{q',p'})}$$

and

$$TT^* f(t) = \int_0^\infty S(t+s)f(s)ds = \int_{-\infty}^0 S(t-s)f(-s)ds$$

and the bound follows as above.

2.2. **Strichartz estimates for the Airy equation.** This section follows Kenig, Ponce and Vega [14]. Scaling shows that the solution to the Airy equation satisfies

$$u(t,x) = \frac{1}{t^{1/3}} \int \mathrm{Ai}(x-y)u(0,y)dy$$

and we obtain the estimates

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}$$

$$\|u(t)\|_{L^\infty} \le ct^{-1/3}\|u_0\|_{L^2}$$

and

$$\||D|^{\frac12} u(t)\|_{L^\infty} \le ct^{-\frac12}\|u_0\|_{L^1}.$$

The Strichartz estimate is more complicated. Here we use complex interpolation to see for $2 < p \le \infty$

$$\|D^{\frac12 - \frac1p} S(t) v\|_{L^p} \le c|t|^{\frac1p - \frac12}\|v\|_{L^{p'}} \tag{2.3}$$

where D^s is defined through the Fourier multiplier. The multiplication on the Fourier side commutes with the evolution, and hence this estimates is equivalent to

$$\|D^{\frac1q} S(t)v\|_{L^p(\mathbb{R})} \le c|t|^{-\frac2q}\|D^{-\frac1q} v\|_{L^{p'}}$$

The Strichartz estimates take the form

Theorem 2.2. *The variation of constants formula defines a function u which satisfies*

$$\partial_t u + u_{xxx} = f, \qquad u(0) = u_0$$

and

$$\|u\|_{C_b(\mathbb{R},L^2)} + \||D|^{\frac1q} u\|_{L^q L^p} \le c\|u(0)\|_{L^2} + \||D|^{-\frac1q} f\|_{L^{q'} L^{p'}}.$$

Proof. It remains to prove (2.3). We claim that it follows from

$$\left| \int |\xi|^{\frac12 + i\sigma} e^{i\xi^3 + i\xi x} d\xi \right| \le C(1+|\sigma|) \tag{2.4}$$

uniformly in x - which has to be understood as oscillatory integral. We apply then complex interpolation with the family of operators

$$\widehat{T_\lambda u_0} = e^{\lambda^2}|D|^{\frac{\lambda}{2}}\widehat{S(t)u_0}$$

for which we easily see that

$$\|T_{i\sigma} u_0\|_{L^2} = e^{-\sigma^2}\|u_0\|_{L^2}$$

and

$$\|T_{i+\sigma} u_0\|_{L^\infty} \le ct^{-1/2}(1+|\sigma|)e^{-\sigma^2}\|u_0\|_{L^1}.$$

Now (2.3) follows by complex interpolation. We turn to (2.4).

There are three cases: $|x| \le 10$, $x \ge 10$ and $x \le -10$. The last one is the hardest since there are large critical points $\pm\xi_c = \sqrt{-x/3}$ in the phase, and we restrict to it. We split the integration into the intervals

$$(-\infty, -\xi_c - |x|^{-1/4}), (-\xi_c - |x|^{-1/4}, \xi_c + |x|^{-1/4}), (-\xi_c + |x|^{-1/4}, -1), (-1,1),$$

$$(1, \xi_c - |x|^{-1/4}), (\xi_c - |x|^{-1/4}, \xi_c + |x|^{-1/4}), (\xi_c + |x|^{-1/4}, \infty)$$

The argument is immediate for the second, the fourth and the sixth integral, which we estimate by $3\xi_c^{1/2}|x|^{-1/4}$. Now

$$\int_{-\infty}^{-\xi_c-|x|^{-1/4}} |\xi|^{\frac{1}{2}+i\sigma} e^{i\xi^3+ix\xi} d\xi = i\int_{-\infty}^{-\xi_c-|x|^{-1/4}} e^{i\xi^3+ix\xi}\frac{d}{d\xi}\frac{|\xi|^{\frac{1}{2}+i\sigma}}{3\xi^2+x}d\xi + \frac{(\xi_c+|x|^{-1/4})^{\frac{1}{2}+i\sigma}}{3(\xi_c+|x|^{-1/4})^2+x}e^{-i(\xi_c+|x|^{-1/4})^3-i(\xi_c+|x|^{-1/4})x}$$

and the direct estimate as for stationary phase gives the result. The largest term (in terms of σ) occurs when the derivative falls on $|\xi|^{\frac{1}{2}+i\sigma}$ - all the others are estimates as when $\sigma = 0$. We recall that

$$3(\xi_c + |x|^{-1/4})^2 + x \sim |x|^{\frac{1}{4}}.$$

□

2.3. **The Kadomtsev-Petviashvili equation.** The symbol is $\xi^3 - \eta^2/\xi$, with gradient

$$\begin{pmatrix} 3\xi^2 + \eta^2/\xi^2 \\ -2\eta/\xi \end{pmatrix}$$

and Hessian matrix

$$\begin{pmatrix} 6\xi - 2\eta^2/\xi^3 & 2\eta/\xi^2 \\ 2\eta/\xi^2 & -2/\xi \end{pmatrix}$$

and Hessian determinant -12.

Lemma 2.3. *The following Strichartz estimate holds*

$$\|u\|_{L^\infty_t L^2_x} + \|u\|_{L^p_t L^q_x} \le c\left(\|u_0\|_{L^2} + \|f\|_{L^{p'}_t L^{q'}_x}\right).$$

The proof is the same (since the same dispersive estimate holds) as for the Schrödinger equation.

2.4. **The (half) wave equation and the Klein-Gordon equation.** Here we only state the result. The proof requires a sharpening of complex interpolation, replacing L^∞ by BMO. The estimates for the wave equation imply that

$$\||D|^{-\frac{d+1}{2}} S(t)v\|_{BMO} \le ct^{-\frac{d-1}{2}}\|v\|_{L^1(\mathbb{R}^d)}$$

which implies

$$\||D|^{-\frac{d+1}{2}(1-\frac{2}{p})} S(t)v\|_{L^p} \le ct^{-\frac{d-1}{2}(1-\frac{2}{p})}\|v\|_{L^{p'}}$$

where the half wave evolution operator $S(t)$ is defined by

$$S(t)v = \mathcal{F}^{-1}(e^{it|\xi|}\hat{v}).$$

As a consequence we obtain

Theorem 2.4. *Let $d \ge 2$. The variation of constants formula defines a function u which satisfies*

$$i\partial_t u + |D|u = f, \qquad u(0) = u_0$$

and

$$\|u\|_{C_b(\mathbb{R},L^2)} + \||D|^{-\frac{d+1}{4}(1-\frac{2}{p})}u\|_{L^q L^p} \le c\|u(0)\|_{L^2} + \||D|^{\frac{d+1}{4}(1-\frac{2}{p})} f\|_{L^{q'} L^{p'}}.$$

where q satisfies $2 < q < \infty$, $2 \le p \le \infty$ and

$$\frac{1}{q} + \frac{d-1}{p} = \frac{d-1}{2}.$$

2.5. **The endpoint Strichartz estimate.** The argument for the endpoint Strichartz estimate depends on a general idea, which we spell out only for the Schrödinger equation

$$iu_t + \Delta u = f \qquad u(0) = u_0$$

for $d \geq 3$. It is due to Keel and Tao [13].

Theorem 2.5. *The solution defined by the variation of constants formula satisfies*

$$\|u\|_{L^\infty_t L^2_x} + \|u\|_{L^2_t L^{\frac{2d}{d-2}}} \leq c \left(\|u_0\|_{L^2} + \|f\|_{L^2_t L_x^{\frac{2d}{d+2}}} \right). \tag{2.5}$$

Before we prove the statement we need a robust estimate for integral operators.

Lemma 2.6 (Schur's lemma)**.** *Let μ and ν be measures,*

$$Tf(x) = \int K(x,y) f(y) d\mu(y)$$

where K satisfies

$$\sup_x \int |K(x,y)| d\mu(y) \leq C, \quad \sup_y \int |K(x,y)) d\nu(y) \leq K.$$

Then

$$\|Tf\|_{L^p(\nu)} \leq C \|f\|_{L^p(\mu)}.$$

Proof. By duality the claim is equivalent to

$$\left| \int f(x) g(y) K(x,y) d\mu(y) d\nu(x) \right| \leq C \|f\|_{L^{p'}(\mu)} \|g\|_{L^p(\mu)}$$

This is obvious for $p = \infty$ and $p = 1$. Hence the operator satisfies the desired bounds on L^1 and L^∞. The claim follows by complex interpolation. □

Proof. We denote by $S(t)$ the Schrödinger group. We first prove

$$\left| \int_{s<t} \langle S(-t) f(t), S(-s) g(s) \rangle \right| \leq c \|f\|_{L^2_t L^{\frac{2d}{d-2}}} \|g\|_{L^2_t L^{\frac{2d}{d-2}}} \tag{2.6}$$

which implies by duality

$$\| \int_{-\infty}^t S(t-s) f(s) \|_{L^2 L^{\frac{2d}{d-2}}} \leq c \|f\|_{L^2_t L^{\frac{2d}{d+2}}}$$

and, by the TT^* argument the full statement.

We define

$$T_j = \int_{t-2^{j+1} < s \leq t - 2^j} \langle S(-s) f(s), S(-t) g(t) \rangle ds dt$$

and claim

$$|T_j| \leq C 2^{-j\beta(p,\tilde p)} \|f\|_{L^2 L^{p'}} \|g\|_{L^2 L^{\tilde p'}} \tag{2.7}$$

for $j \in \mathbb{Z}$, p and $\tilde p$ in a neighborhood of $\frac{2d}{d+2}$ and

$$\beta(p, \tilde p) = \frac{d}{2} - 1 + \frac{d}{2p} + \frac{d}{2\tilde p}.$$

It vanishes for $p = \tilde p = \frac{2d}{d-2}$ as it should.

We set $\tilde t = t 2^{-j}$, $\tilde s = s 2^{-j}$, $\tilde x = 2^{-j/2} x$ and $\tilde y = 2^{-j/2} y$. This transformation of coordinates (which reflects the symmetry) reduces the estimate to the case $j = 0$.

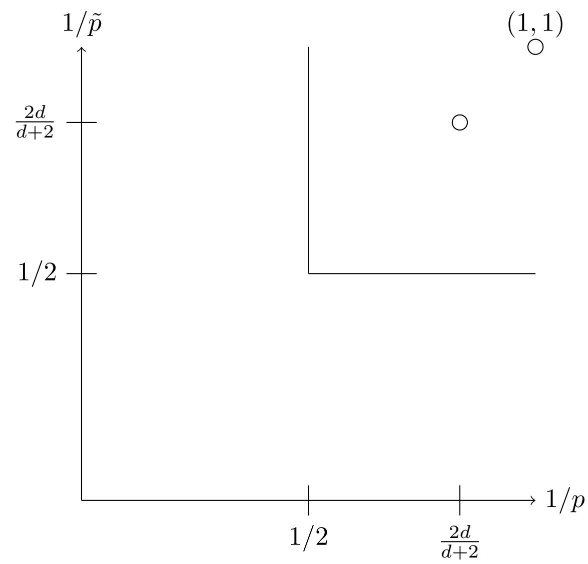
The estimate for $j = 0$

$$|T_0| \le C\|f\|_{L^2L^{p'}}\|g\|_{L^2L^{\tilde p'}} \tag{2.8}$$

holds for

(1) $p = \tilde p = 1$ by the dispersive estimate
(2) $\tilde p = 2$ and $\frac{2d}{d+2} < p' \le 2$
(3) $p = 2$ and $\frac{2d}{d+2} \le \tilde p' \le 2$

Then the estimate (2.7) follows by complex interpolation and duality. It is convenient to draw a diagram



Convex interpolation - this time for $L^{2,p'}$ spaces gives the convex envelope which contains the point $(\frac{2d}{d+2}, \frac{2d}{d+2})$ in its interior.

For the first case (which corresponds to $(1,1)$) observe that by the dispersive estimate, if $t-2 < s < -1$

$$|\langle S(t-s)g(s), f(t)| \le C\|f(t)\|_{L^1}\|g(s)\|_{L^1}$$

Let $h_f(t) = \|f(t)\|_{L^1}$ and $h_g(t) = \|g(t)\|_{L^1}$. Then

$$|T_0(f,g)| \le C\int\int K(t,s)h_g(s)ds h_f(t)dt$$

where $K(t-s) = 1$ if $t-2 < s < t-1$ and 0 otherwise. The first estimate follows by Schur's lemma.

For the second estimate (which corresponds to the horizontal line) we use nonendpoint Strichartz estimate and finally Hölder's inequality to bound

$$\begin{aligned}\left|\int_{s+1}^{s+2}\langle f(t), S(t-s)g(s)\rangle dt\right| \le & \|f\|_{L^{q'p'}([s+1,s+2]\times\mathbb{R}^d}\|S(t-s)g(s)\|_{L^{qp}} \\ \le & C\|f\|_{L^{2,p'}([s+1,s+2]\times\mathbb{R}^d}\|g(s)\|_{L^2}.\end{aligned}$$

where (q,p) is a strict Strichartz pair.

Thus

$$\left| \int_k^{k+1} \int_{t-2}^{t-1} \langle S(-t)f(t), S(-s)g(s)\rangle ds dt \right| \le c\|f\|_{L^{2,p'}([k,k+1]\times\mathbb{R}^d)} \|g\|_{L^{2,2}([k-2,k]\times\mathbb{R}^d)}.$$

The statement follows by summation with respect to k, and the Cauchy-Schwartz inequality with respect to k.

The third estimates follows by the same argument. This completes the estimate (2.7) for $(p, \tilde p)$ close to $(\frac{2d}{d-2}, \frac{2d}{d-2})$.

To make use of the flexility we decompose $f = \sum f_k$, $g = \sum g_k$ such that

$$f_k(t,x) = c_k(t)\chi_{t,k}(x), g_k(t,x)) d_k(t)\tilde\chi_{t,k}(x).$$

We define the decomposition as follows. Given $f : \mathbb{R}^d \to \mathbb{R}$ we define its distribution function

$$\lambda(s) = m^d\{x : |f(x)| > s\}.$$

It is monotonically decreasing and finite for $f \in L^p$. Let s_k be the infimum of all s so that $\lambda(s) < 2^k$ - we allow $s = 0$. We set $c_k = 2^{k/p} s_k$ and

$$\chi_k(x) = c_k^{-1} \begin{cases} f & \text{if } s_k < |f| < s_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$f = \sum c_k \chi_k$$

and, for some $C > 0$

$$C^{-1}\|f\|_{L^p} \le \|(c_k)\|_{l^p} \le C\|f\|_{L^p}$$

which can be seen by comparing to

$$\|f\|_{L^p}^p = p\int m^d(\{|f| > s\}) s^{p-1} ds.$$

By definition

$$m^d(\text{ supp } \chi_k) \le 2^k \qquad |\chi_k| \le 2^{k/p}.$$

We apply this decomposition at every time t with $p = \frac{2d}{d+2}$. Then

$$f = \sum f_k$$

where at most one summand differs from 0.

We apply the first estimate (2.7):

$$\begin{aligned} |T_j(f_k, g_{k'})| \le & c 2^{-\beta(p,\tilde p)} \|f_k\|_{L^{2,p'}} \|g_{k'}\|_{L^{2,\tilde p'}} \\ \le & c 2^{-(\frac{d-2}{2} + \frac{d}{2p} + \frac{d}{2\tilde p}) j + k\left(\frac{1}{p'} + \frac{1}{p} - \frac{d+2}{d}\right)} \|f_k\|_{L^{2,\frac{2d}{d+2}}} \|g_{k'}\|_{L^{2,\frac{2d}{d+2}}} \end{aligned}$$

where the second inequality follows from

$$\|\chi_{t,k}\|_{L^p} \le c 2^{k(\frac{1}{p} - \frac{2d}{d+2})}$$

We optimize p and $\tilde p$. Thus

$$|T_j(f_k, g_{k'})| \lesssim 2^{-\varepsilon(|k-jd/2| + |k'-jd/2|)} \|f_k\|_{L^2 L^{\frac{2d}{d+2}}} \|g_{k'}\|_{L^2 L^{\frac{2d}{d+2}}}$$

for some $\varepsilon > 0$ and e sum with respect to j:

$$\begin{aligned}\sum_j |T_j| \leq & C\sum_k \sum_{k'} (1+|k-k'|)2^{-\varepsilon|k-k'|} \|f_k\|_{L^2 L^{\frac{2d}{d+2}}} \|g_k\|_{L^2 L^{\frac{2}{d}} d+2}. \\ \leq & C\left(\sum_k \|f_k\|^2_{L^2 L^{\frac{2d}{d+2}}}\right)^{1/2} \left(\sum_k \|g_k\|^2_{L^2 L^{\frac{2d}{d+2}}}\right)^{1/2}\end{aligned}$$

by Schur's lemma. By Minkowski's inequality

$$\begin{aligned}\sum_k \int \left(\int_{\mathbb{R}^d} |g_k|^{\frac{2d}{d+2}} dm^d\right)^{\frac{d+2}{d}} dt = & \int \sum_k \left(\int_{\mathbb{R}^d} |g_k|^{\frac{2d}{d+2}} dm^d\right)^{\frac{d+2}{d}} dt \\ \leq & \int \left(\int_{\mathbb{R}^d} \sum_k |g_k|^{\frac{2d}{d+2}} dm^d\right)^{\frac{d+2}{2}} dt \\ = & \|g\|^2_{L^{2,\frac{2d}{d+2}}}\end{aligned}$$

and hence we obtain (2.6).

□

2.6. **Small data solutions to the nonlinear Schrödinger equation.** Most of this section can be found in [4].

We study the initial value problem for initial data $u_0 \in L^2$ for

$$iu_t + \Delta u = \pm |u|^\sigma u \tag{2.9}$$

where $0 \leq \gamma \leq \frac{4}{d-2}$. The case of the plus sign is called defocusing and case of the minus sign is called focusing. At least formally

$$M = \int_{\mathbb{R}^d} |u|^2 dx$$

called mass, and

$$\int_{\mathbb{R}^d} iu\partial_i \bar{u} dx$$

called momentum

$$E = \int_{\mathbb{R}} \frac{1}{2}|\nabla u|^2 \pm \frac{1}{\sigma+2}|u|^{\sigma+2} dx$$

called energy are conserved. For most of this section there is no distinction between the focusing and the defocusing case.

The argument will rely on the Strichartz estimates with $p = q = \frac{2(d+2)}{d}$ and $p' = q' = \frac{2(d+2)}{d+4}$.

The sign of the coefficients is of almost no importance in this section, and we choose + to cover both signs, indicating differences whenever necessary. This section establishes basic schemes which will be used over and over again. Simultaneously it is a warm up the set up and the consequences of the key multilinear estimate. Lateron we will restrict ourselves often to giving the estimates of the nonlinearity, and stating the properties.

It provides also a play ground for stability estimates, qualitative properties, criticality and subcriticality.

2.7. **Initial data in L^2.** Our approach will be based on the Strichartz estimates of Theorem 2.1 with $p = q = \frac{(2(d+2)}{d}$.

$$\|v\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} + \|v\|_{C(\mathbb{R};L^2(\mathbb{R}^d))} \lesssim \|v(0)\|_{L^2(\mathbb{R}^d)} + \|i\partial_t v + \Delta v\|_{L^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)} \tag{2.10}$$

In order to prepare for variants and improvements we assume that there is a space X with

$$X \subset C(\mathbb{R}; L^2(\mathbb{R}^d)) \cap L^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d) \tag{2.11}$$

and

$$\sup_t \|v(t)\|_{L^2} + \|v\|_{L^{\frac{2(d+2)}{d}}} \le c\|v\|_X$$

and

$$\|v\|_X \le c\left(\|v(0)\|_{L^2} + \|i\partial_t v + \Delta v\|_{L^{\frac{2(d+2)}{d+4}}(\mathbb{R}\times\mathbb{R}^d)}\right).$$

Clearly such a space exists: We could define X as the intersection in (2.11), and then the Strichartz estimates ensure that it has the desired properties. The choice of the function space is an important and nontrivial part of studying solutions to many different dispersive equations. Even though we do not need this flexibility here, and even though it complicates the notation a bit we prefer to do it here to indicate possible modifications lateron.

In the sequel we denote by v the solution to the homogeneous equation

$$i\partial_t v + \Delta v = 0, \qquad v(0) = u_0$$

which we can write by the unitary Schrödinger group $S(t)$ as

$$v(t) = S(t)u_0.$$

To approach the question of existence and uniqueness we make the Ansatz $u = v + w$ where v satisfies the linear Schrödinger equation with initial data u_0 , and w satisfies $w(0) = 0$ and

$$\begin{aligned} iw_t + \Delta w =& \chi_{(0,T)}(t)|v+w|^\sigma(v+w) \qquad \text{in } \mathbb{R}\times\mathbb{R}^d \\ w(0,x) =& 0 \qquad \text{in } \mathbb{R}^d \end{aligned} \tag{2.12}$$

where $T \in (0,\infty]$ will be chosen later. We will construct a unique w in X by a fixed point argument. It is obvious that $u = v + w$ is the unique solution up to time T. Then $u = v + w$ is the searched for solution on the time interval $(0, T)$.

We rewrite the problem as a fixed point problem: Given $\tilde{w}$ we write $w = J(\tilde{w})$ where J maps $\tilde{w}$ to the function w which satisfies

$$iw_t + \Delta w = \chi_{(0,T)}(t)|v+\tilde{w}|^\sigma(v+\tilde{w}), \quad w(0) = 0. \tag{2.13}$$

Suppose first that $\frac{2(d+2)}{d+4}(1+\sigma) \ge 2$ and $\sigma \le \frac{4}{d}$. By Hölder's inquality

$$\|f\|^{1+\sigma}_{L^{(1+\sigma)\frac{2(d+2)}{d+4}}(\mathbb{R}^d)} \le \|f\|^{\frac{4-d\sigma}{2}}_{L^2(\mathbb{R}^d)} \|f\|^{\frac{d+2}{2}\sigma-1}_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^d)}$$

Observe that the exponent of $\|f\|_{L^2}$ is nonnegative if $\sigma < \frac{4}{d}$ and it vanishes if $\sigma = \frac{4}{d}$.

If $0 < \frac{2(d+2)}{d+4}(1+\sigma) \le 2$ we estimate again by Hölder's inequality

$$\|f\|^{1+\sigma}_{L^{(1+\sigma)2}(\mathbb{R}^d)} \le \|f\|^{1-\frac{d\sigma}{4}}_{L^2(\mathbb{R}^d)} \|f\|^{(1+\frac{d}{4})\sigma}_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^d)}.$$

In the first case we obtain the space-time estimate

$$\|\chi_{(0,T)}|u|^{1+\sigma}\|_{L^{\frac{2(d+2)}{d+4}}} \le T^{1-\frac{d\sigma}{4}}\|u\|_{L^\infty L^2}^{\frac{4-d\sigma}{2}}\|u\|_{L^{\frac{2(d+2)}{d}}}^{\frac{d+2}{2}\sigma-1} \tag{2.14}$$

and in the second case

$$\|\chi_{(0,T)}|v|^{1+\sigma}\|_{L^1_tL^2_x(\mathbb{R}^d)} \le T^{1-\frac{d\sigma}{4}}\|u\|_{L^\infty L^2}^{1-\frac{d\sigma}{4}}\|u\|_{L^{\frac{2(d+2)}{d}}([0,T]\times\mathbb{R}^d)}^{(1+\frac{d}{4})\sigma}. \tag{2.15}$$

If $\sigma < \frac{4}{d}$ T carries a positive power and we call this situation L^2 subcritical. This power becomes zero if $\sigma = \frac{4}{d}$, which we call L^2 or mass critical.

In the both cases

$$\|J(\tilde w)\|_X \le cT^{1-\frac{d\sigma}{4}}(\|\tilde w\|_X + \|v\|_X)^{1+\sigma}$$

which we complement by the similar estimate

$$\|J(w)-J(\tilde w)\|_X \le cT^{1-\frac{d\sigma}{4}}(\|\tilde w\|_X + \|w\|_X + \|v\|_X)^{\sigma}\|w-\tilde w\|_X.$$

We set up the problem for an application of the contraction mapping principle Let $R=\|v\|_X$. If $\|\tilde w\|_X \le R$ then, for some $c>0$,

$$\|w\|_X \le cT^{1-\frac{d\sigma}{4}}(2R)^{1+\sigma} \le R$$

where the last inequality holds provided

$$T \le (2c(2R)^\sigma)^{-\frac{4}{4-d\sigma}} := T_0$$

which we assume in the sequel. Moreover, if w and $\tilde w$ have norm at most R then

$$\|J(w)-J(\tilde w)\|_X \le cT^{1-\frac{d\sigma}{4}}R^\sigma\|w-\tilde w\|_X$$

We obtain a contraction after decreasing T if necessary.

The critical case requires slightly different arguments, and it yields different conclusions. This time we cannot gain a small power of T and the smallness must have a different source.

In the mass critical case we assume that $\|\chi_{(0,T)}v\|_{L^{\frac{2(d+2}{d}}L^{\frac{2(d+2)}{d}}} \le \varepsilon$ for some small ε.

This is true for all T by Lemma (2.10) if $\|u_0\|_{L^2}$ is sufficiently small. Moreover, for all initial data $u_0 \in L^2$ we have by dominated convergence

$$\|\chi_{(0,T)}v\|_{L^pL^q} \to 0 \qquad \text{as } T \to 0 \tag{2.16}$$

for all Strichartz pairs with $q<\infty$.

It is obvious from the argument above (where we replace $\|\chi_{(0,T)}v\|_X$ by $\|\chi_{(0,T)}v\|_{L^{\frac{2(d+2)}{2}}}$ for the mass critical case) that the iteration argument applies if ε is sufficiently small. We obtain local existence under the smallness assumption, and hence global existence provided the initial data are sufficiently small.

We collect the results in a theorem.

Theorem 2.7. *There exists $\varepsilon > 0$ such that the following is true. Suppose that $0<\sigma\le\frac{4}{d}$, $u_0\in L^2$ and*

$$T^{1-\frac{d\sigma}{4}}\|\chi_Tv\|_X^\sigma < \varepsilon.$$

resp. $\sigma = \frac{4}{d}$ and

$$\|\chi_Tv\|^\sigma_{L^{\frac{2(d+2)}{2d}}(\mathbb{R}\times\mathbb{R}^d)} < \varepsilon.$$

Then there is a unique solution in X up to time T which satisfies

$$\|u-v\|_X \lesssim T^{1-\frac{d\sigma}{4}}\|v\|_X^{1+\sigma} \tag{2.17}$$

resp, if $\sigma = \frac{4}{d}$,

$$\|u - v\|_X \lesssim T^{1-\frac{d\sigma}{4}} \|v\|^{1+\sigma}_{L^{\frac{2(d+2)}{d}}} \tag{2.18}$$

There is a unique global solution

$$u \in L^{\frac{2(d+2)}{d}}((-T,T) \times \mathbb{R}^d) \cap C((-T,T); L^2(\mathbb{R}^d))$$

for all T *if either* $0 \le \sigma < \frac{d}{4}$*, or, if* $\|u_0\|_{L^2} \le \varepsilon$ *and* $\sigma = \frac{d}{4}$*. In the last case we have* (2.18) *with* $T = \infty$*. If* $0 \le k < 1 + \sigma$ *then*

$$(u_0 \to u) \in C^k(L^2(\mathbb{R}^d); X)$$

There is a stability estimate. Suppose that $\tilde{u} \in X$ *satisfies*

$$T^{1-\frac{d\sigma}{4}} \|\tilde{u}\|_X < \varepsilon$$

$$\|\tilde{u} - u_0\|_{L^2} + \|i\partial_t \tilde{u} + \Delta \tilde{u} - |\tilde{u}|^\sigma \tilde{u}\|_{L^{\frac{2(d+2)}{d+4}}} < \varepsilon.$$

Then there exists a unique solution up to time T *with*

$$\|u - \tilde{u}\|_X \le c\left(\|\tilde{u} - u_0\|_{L^2} + \|i\partial_t \tilde{u} + \Delta \tilde{u} - |\tilde{u}|^\sigma \tilde{u}\|_{L^{\frac{2(d+2)}{d+4}}} \right). \tag{2.19}$$

If $\sigma = \frac{4}{d}$ *it suffices to require*

$$\|\chi_{(0,T)} \tilde{u}\|_{L^{\frac{2(d+2)}{d}}} < \varepsilon$$

Proof. Local existence in the subcritical case has been shown above. The fixed point formulation leads to existence via the contraction mapping theorem on a time interval whose length depends only on $\|u_0\|_{L^2}$. We claim that the L^2 norm (mass) is conserved. Indeed, for sufficiently regular and decaying $\tilde{u} = v + \tilde{w}$ and $u = v + w$ we have

$$\frac{1}{2}\|u(t)\|^2_{L^2} = \frac{1}{2}\|u_0\|^2_{L^2} + \text{ real} i \int_{(0,t)\times\mathbb{R}^d} |\tilde{u}|^\sigma \tilde{u}\bar{u} dx dt$$

which remains true for general $\tilde{u}$ and initial data by an approximation argument. By then it also holds for the fixed point, for which the second term on the right hand side is the real part of something purely imaginary.

Thus we can extend the solution to a global solution in the subcritical case.

It follows from the construction by the contraction mapping principle that the solution depends Lipschitz continuously on the initial data.

The map

$$L^{\frac{2(d+2)}{d}}(\mathbb{R} \times \mathbb{R}^d) \ni w \to \chi_{(0,T)}|w|^\sigma w \in L^{\frac{2(d+2)}{d+4}}(\mathbb{R} \times \mathbb{R}^d)$$

is k times continuously differentiable for $k < 1 + \sigma$, and $\sigma \le \frac{4}{d}$.

Thus J is k times continuously Frechet differentiable. Moreover, by the very same estimates as for the contraction the derivative of J with respect to $\tilde{w}$ is invertible, and by the implicit function theorem from the initial data to the solution is k times continuously differentiable. Checking the norms implies the stability estimate.

□

We also have

$$\lim_{T\to\infty} \|\chi_{(T,\infty)} v\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} = 0.$$

Suppose that $u \in X$ is a solution for $T = \infty$ and $\sigma = \frac{4}{d}$. One can deduce that the limit

$$\lim_{t\to\infty} S(-t)u(t)$$

exists in L^2. Let w_0 be this limit, and w the solution to the homogeneous equation with initial data w_0. Then the convergence statement can be formulated as

$$\lim_{t\to\infty} \|u(t) - w(t)\|_{L^2} = 0.$$

This is called scattering.

2.8. **Initial data in $\dot H^1$ for $d \geq 3$.** Consider

$$iu_t + \Delta u = \pm |u|^\sigma u \tag{2.20}$$

with initial data $u_0 \in \dot H^1$, by which we mean the space with the norm $\| \, |\nabla u_0| \, \|_{L^2}$. We want to use Strichartz spaces for the derivative and we define the function spaces X by

$$\|u\|_X := \sup_t \|\nabla u(t)\|_{L^2} + \|\nabla u\|_{L^{\frac{2(d+2)}{d}}}.$$

Then the Strichartz estimate 2.10 combined with Sobolev's estimate gives

$$\|u\|_X \leq c \left(\|\nabla u_0\|_{L^2} + \|\nabla f\|_{L^{\frac{2(d+2)}{d+4}}} \right)$$

for a solution u to the inhomogeneous linear problem.

Then, if $\sigma \leq \frac{4}{d-2}$, by Hölder's and Sobolev's inequality

$$\|\nabla |f|^\sigma f\|_{L^{\frac{2(d+2)}{d+4}}(\mathbb{R}^d)} \lesssim \|f\|^{\frac{4-(d-2)\sigma}{2}}_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \|\nabla f\|^{-1+\frac{4-d}{2}\sigma}_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^d)}$$

provided σ is not too small. For small σ we argue as for the case of L^2. We obtain in both cases

$$\|J(w)\|_X \lesssim T^{1-\frac{(d-2)\sigma}{4}} \left(\|v\|_X + \|w\|_X\right)^{1+\sigma}, \tag{2.21}$$

and, checking the same argument for differences,

(2.22)

$$\|J(w^2) - J(w^1)\|_{L^{\frac{2(d+2)}{d}}} + \|J(w^2) - J(w^1)\|_{L^\infty L^2} \lesssim T^{1-\frac{(d-2)\sigma}{4}} \left(\|v\|_X + \|w^1\|_X + \|w^2\|_X\right)^\sigma \times \left(\|w^2 - w^1\|_{L^{\frac{2(d+2)}{d}}} + \|w^2 - w^1\|_{L^\infty L^2}\right)$$

Theorem 2.8 (Local existence and uniqueness in energy space)**.** *Suppose that $0 < \sigma \leq \frac{4}{d-2}$. There exists $\varepsilon > 0$ such that the following is true. Let v be the solution to the homogeneous linear Schroedinger equation. Suppose that*

$$T^{1-\frac{(d-2)\sigma}{4}} \|v\|_X^\sigma \leq \varepsilon$$

Then there exists a unique solution $u = v + w$ with

$$\|\nabla w\|_{L^\infty L^2} + \|\nabla w\|_{L^{\frac{2(d+2)}{d}}} \lesssim T^{1-\frac{(d-2)\sigma}{4}} \|v\|_X^{1+\sigma}.$$

Again we may replace $\|v\|_X$ by $\|\chi_{0,T} \nabla v\|_{L^{\frac{2(d+2)}{d}}}$. In the defocusing case the solution is global if $\sigma < \frac{4}{d-2}$. In the energy critical case $\sigma = \frac{4}{d-2}$ there is global existence for small data, and local existence for all data in $\dot H^1$.

Proof. Again we characterize the solution as the fixed point of the same map as above, but now with respect to the norm X. By (2.21) we obtain a map of a closed ball in X to itself, but a contraction only in the metric of $L^{\frac{2(d+2)}{d}}$ in a ball in X- at least for large space dimensions and small σ. We change the space X slightly by replacing $C(\mathbb{R};L^2)$ by $L^\infty(\mathbb{R};L^2)$. We claim that sequences which are bounded in X and converge in $L^{\frac{2(d+2)}{d}}$ have a limit in X. There is a weak* converging subsequence in X, and the limits have to coincide.

It is not hard to complete the argument for initial data additionally in $L^2(\mathbb{R}^d)$: then $v \in L^{\frac{2(d+2)}{2}}$, and this remains true for the fixed point map. In general we define iteratively $v_{j+1} = J(v_j)$. We claim that there exists j so that

$$v_{j+1} - v_j \in L^{\frac{2(d+2)}{d}}(\mathbb{R} \times \mathbb{R}^d).$$

The contraction argument then completes the proof. This argument gives uniqueness in the set

$$v_j + X \cap L^{\frac{2(d+2)}{d}}(\mathbb{R} \times \mathbb{R}^d).$$

The proof of the claim is technical and omitted.

The remaining arguments are adaptations of similar arguments in Theorem 2.7. □

2.9. **Initial data in $H^1(\mathbb{R}^d)$.** In this case we combine the arguments. We obtain global wellposedness in the defocussing subcritical case $\sigma < \frac{d}{4}$, local existence in the subcritical and the critical case ($\sigma \le \frac{d}{4}$) and global existence in the critical case $\sigma = \frac{d}{4}$ and small initial data.

3. Functions of bounded p variation

The study of p variation of functions of one variable has a long history. Function of bounded p variation have been studied by Wiener in [31]. The generalization of the Riemann-Stieltjes integral to functions of bounded p variation against the deriative of a function of bounded q variation $1/p + 1/q > 1$ is due to Young [32]. Much later Lyons developed his theory of rough path [21] and [22], building on Young's ideas, but going much further.

In parallel D. Tataru realized that the spaces of bounded p variation, and their close relatives, the U^p spaces, allow a powerful sharping of Bourgain's technique of function spaces adapted to the dispersive equation at hand. These ideas were applied for the first time in the work of the author and D. Tataru in [16]. Since then there has been a number of questions in dispersive equations where these function spaces have been used. For example they play a crucial role in [17], but there they could probably be replaced by Bourgain's Fourier restriction spaces $X^{s,b}$. On the other hand, for wellposedness for the Kadomtsev-Petviashvili II in a critical function space (see [10]) the $X^{s,b}$ spaces seem to be insufficient. The theory of the spaces U^p and V^p and some of their basic properties like duality and logarithmic interpolation have been worked out for the first time in [10]. The development in stochastic differential equations and dispersive equations has been largely independent.

It is the purpose of this section to introduce the spaces U^p and V^p. We will show that there is a very pleasant treatment of ODE's

$$\dot{y} = f(y)\dot{x}$$

if f is Lipschitz continuous and $x \in U^2 \subset V^2$ - the endpoint case of Lyons related treatment of the case $x \in V^p$, $p < 2$ [21]. One may hope that a similar approach works for differential equations with rough paths.

In later chapters we use U^p and V^p to study wellposedness questions for several dispersive PDEs, where we select a number of relevant and representative problems.

In the sequel $p \in [1, \infty]$. Unless explicitly stated otherwise we consider $p \in (1, \infty)$.

3.1. Bounded p variation.

Definition 3.1. *Let I be an interval, X a Banach space, $1 \le p < \infty$ and $f : I \to X$. We define*

$$\omega_p(v, I) := \sup_{t_i \in I, t_1 < t_2 < \dots < t_n} \left(\sum_{i=1}^{n-1} \|v(t_{i+1}) - v(t_i)\|_X^p \right)^{1/p} \in [0, \infty].$$

There are obvious properties. The function $t \to \omega_p(v, [a, t))$ is monotonically increasing. The same is true if we consider closed or open intervals.

Lemma 3.2. *Suppose that $a < b < c$. Then*

$$\omega_p(v, [a, b)) \le \omega_p(v, [a, c)) \le 2^{1-1/p} \Big(\omega_p(v, [a, b]) + \omega_p(v, [b, c)) \Big).$$

Proof. Consider a partition τ. If b is a point of τ then the p-th power of the τ variation in the large interval is the sum of the p powers of the parts. If not we add the point b. This increases the right hand side of the second inequality, and it may decrease or increase the left hand side. The factor $2^{1-1/p}$ follows from

$$|a+b|^p \le 2^{p-1}(|a|^p + |b|^p).$$

□

The p variation can sometimes be explicitly estimated.

Lemma 3.3. *For bounded monotone functions we have*

$$\|v\|_{V^p} = \sup v - \inf v.$$

We denote by $\dot C^s(I)$ the homogeneous Hölder norm:

$$\|f\|_{\dot C^s(I)} = \sup_{t\neq\tau} \frac{|u(t)-u(\tau)|}{|t-\tau|^s}.$$

Lemma 3.4. *We have*

$$\omega_p(v,(a,b)) \le \|v\|_{\dot C^{1/p}} (b-a)^{1/p}.$$

Suppose that

$$\omega_p(v,(a,b)) < \infty.$$

Then v has left and right limits at every point. The expression is invariant with respect to continuous monotone coordinate changes. Moreover

$$\omega_p(\lambda v,(a,b)) = |\lambda|\omega_p(v,(a,b)),$$

$$\omega_p(v+w,(a,b)) \le \omega_p(v,(a,b)) + \omega_p(w,(a,b)).$$

Proof. Let $t_0 < t_1 < \dots t_N$. Then

$$\sum_j \|v(t_{i+1}) - v(t_i)\|_X^p \le \sum_i (t_{i+1}-t_i)\|v\|_{\dot C^{1/p}}.$$

The other statement follow from a straightforward calculation. □

The p variation is continuous at points where v is continuous, provided the p variation is finite.

Lemma 3.5. *Suppose that $\omega_p(v,[a,b)) < \infty$ and v is continuous at $c \in [a,b)$. Then*

$$\lim_{t\to c} \omega_p(v,[a,t)) = \omega_p(v,[a,c]).$$

Proof. Suppose that

$$\lim_{t\to b,\, t>a} \omega(v,(a,t)) - \omega(v,(a,b)) = 2\delta > 0.$$

Then there is a sequence of points $c < t_1 < t_2 \dots t_n < b$ with

$$\sum \|v(t_{i+1}) - v(t_i)\|_X^p \ge \frac{\delta^p}{p\|v\|_{sup}^{p-1}}.$$

Similarly there is such a sequence in (c,t_1) and recursively we get an arbitrary large number of such sequences. Putting N of them together we see that

$$\omega_p(v,(c,b)) \ge Nc\delta$$

which would bound N. This is a contradiction. Similarly we argue for the limit from below. □

3.2. **Generalities.** We introduce and study functions from an interval $[a, b)$ to $\mathbb{R}$, $\mathbb{R}^n$, a Hilbert space or a Banach space X, and spaces of such functions which are invariant under continuous monotone reparametrizations of the interval. For the most part of this section there are no more than the obvious modifications when considering Banach space valued functions.

We call a function f ruled function if at every point (including the endpoints, which may be $\pm\infty$) left and right limits exist. The set of ruled functions is closed with respect to uniform convergence. We denote the Banach space of ruled functions equipped with the supremum norm by $\mathcal{R}$.

A partition τ of $[a, b)$ is a strictly increasing finite sequence

$$a < t_1 < t_2 < \cdots < t_{n+1} < b$$

where we allow $b = \infty$ and $a = -\infty$. A step function is a function f for which there exists a partition so that f is constant on every interval (a, t_1), (t_i, t_{i+1}) and (t_n, b). We do not require that the value at a point coincides with the limit from either side. Step functions are dense in $\mathcal{R}$ (Aumann [1], Dieudonne [6]). We denote the set of step functions by $\mathcal{S}$.

Let $\mathcal{R}_{rc}$ be the closed subset of $\mathcal{R}$ of right continuous functions f with $\lim_{t\to a} f(t) = 0$. Similarly, if $A \subset \mathcal{R}$ we denote by A_{rc} the intersection with $\mathcal{R}_{rc}$.

The step functions

$$f_t = \chi_{[t,b)}$$

satisfy

$$\|f_t - f_s\|_{\sup} = \begin{cases} 1 & \text{if } s \neq t \\ 0 & \text{if } s = t. \end{cases} \tag{3.1}$$

We will study Banach spaces A most of which contain the right continuous step functions $\mathcal{S}_{rc}$, and which embed into $\mathcal{R}$. Moreover we will always have

$$1 \le \|f_t - f_s\|_A \le 2 \tag{3.2}$$

and hence none of those spaces is separable.

It will be convenient to extend every function on $[a, b)$ by zero to $[a, b]$, i.e. we will always set $f(b) = 0$, even if $a = -\infty$ or $b = \infty$.

Definition 3.6. *For $f \in \mathcal{R}$ and a partition*

$$\tau = (t_1, t_2 \ldots t_n), \quad a < t_1 < t_2 < t_3 \cdots < t_n < b$$

we define (denoting the limit from the right by $f(t+)$)

$$f_\tau(t) = \begin{cases} f(t) & \text{if } t = t_j \quad \text{for a } j \\ f(a+) & \text{if } a < t < t_1 \\ f(t_i+) & \text{if } t_i < t < t_{i+1} \\ f(t_n+) & \text{if } t_n < t \end{cases}$$

We observe that f_τ is a step function, and it is right continuous if f is right continuous.

3.3. **The spaces V^p and U^p.** In this subsection we consider functions on (a, b) where we allow the cases $a = -\infty$ and $b = \infty$.

Definition 3.7. *Let X be a Banach space , $1 \le p < \infty$ and $v : (a, b) \to X$. We define*

$$\|v\|_{V^p((a,b),X)} = \max\{\|v\|_{sup}, \omega_p(v, (a, b))\}.$$

Let $V^p = V^p((a,b)) = V^p(X) = V^p((a,b);X)$ be the set of all functions for which this expression is finite. We omit the interval and/or the Banach space in the notation when this seems appropriate.

The interval will usually be of minor importance. The following properties are immediate:

(1) $V^p(I)$ is closed with respect to this norm and hence $V^p(I)$ is a Banach subspace of $\mathcal{R}$. Moreover $V^p_{rc}(I)$ is a closed subspace.
(2) We set $V^\infty = \mathcal{R}$ with $\|.\|_{V^\infty} = \|.\|_{sup}$.
(3) If $1 \le p \le q \le \infty$ then
$$\|v\|_{V^q} \le \|v\|_{V^p}.$$
(4) Let X_i be Banach spaces, $T : X_1 \times X_2 \to X_3$ a bounded bilinear operator, $v \in V^p(X_1)$ and $w \in V^p(X_2)$. Then $T(v,w) \in V^p(X_3)$ and
$$\|T(v,w)\|_{V^p(X_3)} \le 2\|T\|\|v\|_{V^p(X_1)}\|w\|_{V^p(X_2)}.$$
(5) We embed $V^p((a,b))$ into $V^p(\mathbb{R})$ by extending v by 0.
(6) The space V^1 has some additional structure: Every bounded monotone function is in V^1, and functions in V^1 can be written as the difference of two bounded monotone functions.

The space of bounded p variation is build on the sequence space l^p. We may also replace it by the weak space l^p_w, with
$$\|(a_j)\|_{l^p_w} = \sup_{\lambda>0} \lambda(\#\{j : |a_j| > \lambda\})^{\frac{1}{p}}.$$
This does not satisfy the triangle inequality, but if $p > 1$ there is an equivalent norm, which makes l^p_w a Banach space. We set $l^\infty_w = l^\infty$.

Definition 3.8. *Let $1 \le p < \infty$. The weak V^p_w space consists of all functions such that*
$$\|v\|_{V^p_w} = \max\{\sup_{t_1<\cdots<t_n} \|(v(t_{i+1}) - v(t_i))_{1\le i\le n-1}\|_{l^p_w}, \|v\|_{sup}\}$$
is finite.

By Tschebycheff's inequality
$$\|v\|_{V^p_w} \le \|v\|_{V^p}.$$

The spaces of bounded p variation are of considerable importance in probability and harmonic analysis. We shall see that V^p is the dual space of a space U^q, $1/p + 1/q = 1$, $1 < p < \infty$, with a duality pairing closely related to the Stieltjes integral, and its variant, the Young integral [32].

Definition 3.9. *A p-atom a is a step function in $\mathcal{S}_{rc}$,*
$$a(t) = \sum_{i=1}^n \phi_i \chi_{[t_i,t_{i+1})}(t)$$
where $\tau = (t_1 \dots t_n)$ is a partition, $t_{n+1} = b$, with $\sum |\phi_i|^p \le 1$. A p-atom a is called a strict p atom if
$$\max \|\phi_i\|_X (\#\tau)^{1/p} \le 1.$$
It is important that atoms are right continuous, zero in a neighborhood of a, but the limit as $t \to b$ may be different from 0.

Let a_j be a sequence of atoms and λ_j a summable sequence. Then

$$u = \sum \lambda_j a_j$$

is a U^p function. This is well defined sind the right hand side converges in $\mathcal{R}$. We define

$$\|u\|_{U^p} := \inf \Big\{ \sum |\lambda_j| : u = \sum \lambda_j a_j \Big\}.$$

The strict space U^p_{strict} is defined in the same fashion using strict p atoms.

We collect a number of elementary properties.

(1) If a is a p-atom then $\|a\|_{U^p} \le 1$. In general the norm of an atom is less than 1. Determining the norm of an atom is propably a difficult task.
(2) Functions in U^p are continuous from the right. The limit as $t \to a$ vanishes.
(3) The expression $\|.\|_{U^p}$ defines a norm on U^p, and U^p is closed with respect to this norm. Moreover $U^p \subset \mathcal{R}_{rc}$ is a subspace with $\|.\|_{sup} \le \|.\|_{U^p}$.
(4) If $p < q$ then $U^p \subset U^q$ and

$$\|u\|_{U^q} \le \|u\|_{U^p}$$

(5) If $1 \le p < \infty$ then for all $u \in U^p$

$$\|u\|_{V^p} \le 2^{1/p} \|u\|_{U^p}$$

(6) Let Y be a Banach space, and let the linear operator $T : \mathcal{S}_{rc} \to Y$ satisfy

$$\|Ta\|_Y \le C \tag{3.3}$$

for every p atom. Then T has a unique extension to a bounded linear operator from U^p to Y which satisfies

$$\|Tf\|_Y \le C\|f\|_{U^p}. \tag{3.4}$$

(7) Let X_i be Banach spaces, $T : X_1 \times X_2 \to X_3$ a bounded bilinear operator, $v \in U^p(X_1)$ and $w \in U^p(X_2)$. Then $T(v, w) \in U^p(X_3)$ and

$$\|T(v, w)\|_{U^p(X_3)} \le 2\|T\|\|v\|_{U^p(X_1)}\|w\|_{U^p(X_2)}.$$

(8) We consider $U^p([a, b))$ in the same way as subspace of $U^p(\mathbb{R})$ as for V^p.

3.4. **A decomposition and logarithmic interpolation.** The following decomposition is crucial for most of the following. It is related to Young's generalization of the Stieltjes integral, and it deals with a crucial point in the theory. We denote the number of points in a partition τ by $\#\tau$.

Lemma 3.10. *There exists $\delta > 0$ such that for v right continuous with $\|v\|_{V^p_w} = \delta$ there are strict p atoms a_i with*

$$\|a_j(t)\|_{sup} \le 2^{1-j} \qquad \text{and} \qquad \#\tau_j \le 2^{jp}$$

such that (in the sense of uniform convergence)

$$v = \sum a_j.$$

Proof. We set $v_0 = v$, and we search for a recursive decomposition with

$$v_j = a_j + v_{j+1}$$

such that

$$\|v_j\|_{sup} \le 2^{-j}, \|a_j\|_{sup} \le 2^{-j}$$

and, with τ_j the partition related to a_j,

$$\#\tau_j \le \kappa 2^{pj}.$$

Suppose we have constructed v_i for $i \le j$ and a_i for $i \le j-1$. We construct the a_j, which also defines v_{j+1}. We choose the unique partition τ so that

$$\|v_j(t)\|_X < 2^{-1-j} \qquad \text{in } [a,t_1), \qquad \|v_j(t_1)\|_X \ge 2^{-1-j},$$
$$\|v_j(t) - v_j(t_i)\|_X < 2^{-1-j} \quad \text{in } t \in [t_i, t_{i+1})$$

and

$$\|v_j(t_{i+1}) - v_j(t_i)\|_X \ge 2^{-1-j}.$$

We define a_j as the step function adapted to the partition τ_j (recall Definition 3.6)

$$a_j = (v_j)_\tau$$

Then, by construction,

$$\|a_j\|_{sup} \le \|v_j\|_{sup} \le 2^{-j},$$
$$\|v_{j+1}\|_{sup} \le 2^{-1-j}$$

and since either $(t_j, t_{j+1}]$ contains no points of an earlier partition, in which case we estimate the sum of these differences using the V^p_w norm of v, or it does, and then we simply add the number of those terms, and iterate

$$\begin{aligned} \#\tau_j \le & \|v\|^p_{V^p_w} 2^{jp} + \sum_{i=0}^{j-1} \#\tau_i \\ \le & \|v\|^p_{V^p_w} \sum_{i=0}^{j} (j+1-i) 2^{ip} \\ \le & c_p \|v\|^p_{V^p_w} 2^{jp} \end{aligned} \tag{3.5}$$

We choose $\delta = c_p^{-1/p}$. □

There are a number of simple interesting and useful consequences.

Lemma 3.11. *Let $1 < p < q < \infty$. There exists $\kappa > 0$, depending only on p and q, such that for all $v \in V^p_{w,rc}$ and $M \ge 1$ there exist $u \in U^p_{strict}$ and $w \in U^q_{strict}$ with*

$$v = u + w$$

and

$$\frac{\kappa}{M} \|u\|_{V^p_{strict}} + e^M \|w\|_{U^q_{strict}} \le \|v\|_{V^p_w}.$$

Observe that we may replace U^p_{strict} by U^p (since $U^p_{strict} \subset U^p$) and V^p_w by V^p (since $V^p \subset V^p_w$).

Proof. Multiplying v by a constant we may assume that $\|v\|_{V^p_w} = \delta$ for some $\kappa > 0$ to be chosen below. Using the notation of Lemma 3.10 and setting $u = \sum_{j=j_0}^m a_j$ we have

$$\|u\|_{U^p_{strict}} \le m.$$

By construction $2^{j(1-p/q)} a_j$ is a strict q atom and hence, with $w = \sum_{j=m+1}^\infty a_j$,

$$\|w\|_{U^q_{strict}} \le \sum_{j=m+1}^{\infty} \|a_j\|_{U^q_{st}} \le 2^{(\frac{p}{q}-1)m}.$$

Given M we choose m so that

$$M \le (\ln 2)(1-\frac{p}{q})m + 2\ln 2 - \ln\delta \le M+2.$$

Then

$$\|u\|_{U^p_{strict}} \le m \le (1-\frac{p}{q}))^{-1}(M+2)$$

and

$$\|w\|_{U^p_{strict}} \le \delta e^{-M}/4.$$

The assertion follows if we choose

$$\delta \le (1-\frac{p}{q})^2/3.$$

□

We obtain the following embedding

Lemma 3.12. *Let $1<p<q<\infty$. Then*

$$V^p_{rc} \subset V^p_{w,rc} \subset U^q_{strict} \subset U^q.$$

Proof. Apply Lemma 3.11 with $M=1$. □

3.5. **Duality and the Riemann-Stieltjes integral.** The Riemann-Stieltjes integral defines

$$\int f dg = \int f g_t dt$$

for $f \in \mathcal{R}$ and $g \in V^1$. If f or $g \in \mathcal{S}_{rc}$ then, with the obvious partition,

$$\int f g_t dt = \sum f(t_i)(g(t_i)-g(t_{i-1})). \tag{3.6}$$

We take this formula as our starting point for a similar integral for $f \in V^p$ and $g \in U^q$, for $1/p+1/q=1$, $q \ge 1$. Results become much cleaner when we use an equivalent norm in V^p,

$$\|v\|_{V^p} = \sup_{a<t_1\dots t_n<b} \left(\sum_{j=1}^{n-1} |v(t_{j+1})-v(t_j)|^p + |v(t_n)|^p \right)^{1/p}$$

which we do in the sequel. We also set $v(b)=0$ and, for any partition, $t_{n+1}=b$.

Theorem 3.13. *There is a unique continuous bilinear map*

$$B : U^q(X) \times V^p(X^*) \to \mathbb{R}$$

which satisfies (with $t_0=a$ and $u(t_0)=0$)

$$B(u,v) = \sum_{i=1}^n v(t_i)(u(t_i)-u(t_{i-1}))$$

for $v \in V^p$ with associated partition $(t_1,\dots t_n)$ and $v(t_i)(.)$ the evaluation of $v(t_i) \in X^*$ on the argument in X, and

$$|B(u,v)| \le \|u\|_{U^q(X)} \|v\|_{V^p(X^*)}. \tag{3.7}$$

The map

$$V^p(X^*) \ni v \to (u \to B(u,v)) \in (U^q(X))^*$$

is a surjective isometry if $1 \le q < \infty$*. Moreover*

$$\|v\|_{V^p(X^*)} = \sup_{u \in U^q(X), \|u\|_{U^q(X)}=1} B(u,v) = \sup_{a \text{ is a } q-atom} B(a,v). \tag{3.8}$$

The same statements are true if we replace U^p *by* U^p_{strict} *and* V^q *by* V^q_w*.*

Proof. Let $v \in V^p$. The expression

$$F_v(u) = \sum_{i=1}^n (u(t_i) - u(t_{i-1}))v(t_i) = -\sum_{i=1}^n u(t_i)(v(t_{i+1}) - v(t_i))$$

is clearly defined for $v \in V^q$ and $u \in \mathcal{S}_{rc}$ with partition $\tau = (t_i)$ - recall that $u(a) = v(b) = 0$. The product is an abuse of notation for the duality pairing between X and X^* which we supress in the notation. The map is linear in v and u and satisfies for every atom (by Hölder's inequality, and using the right hand side of the equation for $F_v(u)$)

$$|F_v(a)| \le \|v\|_{V^p}.$$

Existence of a unique extension to U^q follows from this estimate and (3.4). Linearity in v and estimate (3.7) are immediate consequences. Clearly B defines a map from V^p to the dual of U^q with norm at most 1. Let us prove that it defines an isometry and choose $v \in V^p$, $\varepsilon > 0$, and a partition $t_0 < t_1 < \cdots < t_n$ with

$$\|v\|_{V^p} \le \left(\sum_{j=1}^n \|v(t_{j+1}) - v(t_j)\|^p_{X^*} \right)^{1/p} + \varepsilon.$$

Here we set again $t_{n+1} = b$ and $v(b) = 0$. We choose $x_i \in X$ of norm 1 with

$$(v(t_{i+1}) - v(t_i))(x_i) \ge (1-\varepsilon)\|\|v(t_{i+1}) - v(t_i)\|_{X^*}$$

and

$$\phi_j := \mu\|v(t_{j+1}) - v(t_j)\|^{p-1}_{X^*} x_j$$

where $\mu = \|v\|^{1-p}_{V^p}$. Then

$$\sum_{j=1}^n \|\phi_j\|^{p'}_x \le \mu^{-p} \sum_{j=1}^n \|v(t_{j+1}) - v(t_j)\|^p_{X^*} \le 1.$$

Thus the partition and the ϕ_j define an atom a, and

$$\|v\|_{V^p} \ge B(a,v) - C\varepsilon.$$

Thus the map is an isometry. We turn to surjectivity. Let $F \in (U^q)^*$ and define the element $v(t) \in X^*$ by

$$v(t)(x) := F(x\chi_{[t,\infty)}) \qquad \text{for } x \in X.$$

Let a be an atom. Then

$$F(a) = \sum F(\phi_i \chi_{[t_i,b)}) - F(\phi_i \chi_{[t_{i+1},b)}) = -\sum \phi_i (v(t_{i+1}) - v(t_i)).$$

By the previous estimate

$$\|v\|_{V^p} \le \|F\|_{(U^q)^*}$$

and

$$B(a,v) = F(a)$$

for all atoms a. Hence both sides coincide on U^q. The remaining claims are simple consequences. $\square$

The previous results show that $U^p \subset V^p_{rc}$, and both spaces are very close. They are, however, not equal. The following example goes back to Young [32] with the same intention, but in a slightly different context.

Lemma 3.14. *Let ϕ be a smooth function with compact support, $1 < q < \infty$. Then*

$$u_q(t) = \phi(x) \sum_{j=1}^{\infty} 2^{-j/q} \cos(2^j t) \in V^q_{rc}$$

but not in U^q.

Proof. Let p ne the Hölder dual exponent of q and

$$v_p^N(t) = \phi \sum_{j=1}^{N} 2^{-j/p} \sin(2^j t).$$

where we allow $N = \infty$. Then, with $M = [\ln_2(|t-s|)]$, [ ] the Gauss bracket,

$$\begin{aligned} |v_p^N(t) - v_p^N(s)| \le & \sum_{j=1}^{M} 2^{-j/p} |\phi(t) \sin(2^j t) - \phi(s) \sin(2^j s)| + c_1 \sum_{j=M+1}^{N} 2^{-j/p} \\ \le & c_2 \left(\sum_{j=1}^{M} 2^{-j/p+j} |t-s| + 2^{-j/M} \right) \\ \le & c_3 \left(2^{-M/p+M} |t-s| + 2^{-j/M} \right) \\ \le & c_4 |t-s|^{\frac{1}{p}} \end{aligned}$$

and hence, by Lemma 3.4

$$\sup_N \|V_p^N\|_{V^p} < \infty.$$

Thus $u_q \in V^q_{rc}$ for the Hölder dual exponent q with $\frac{1}{p} + \frac{1}{q} = 1$. Now, assuming that $u_q \in U^q$, we claim

$$\|u_q\|_{U^p} \|v_p^N\|_{V^q} \ge \left| \int (u_q^\infty)' v_p^N \, dx \right| = N/2 \int \phi^2 dx + O(1) \tag{3.9}$$

which is unbounded, hence a contradiction and $V^q_{rc} \ni u_q^\infty \notin U^q$. Hence it remains to verify (3.9). We expand both factors in the integral and claim for $j \neq l$ by stationary phase

$$\left| \int \phi(t) 2^{-j/p - l/q} \cos(2^j t) (\phi(t) \sin(2^l t))' dt \right| \le c_M 2^{-j} |2^j - 2^l|^{-M}$$

for every $M \in \mathbb{N}$. Thus

$$\sum_{j \neq l, l \le N} \left| \int \int \phi(t) 2^{-j/p - l/q} \cos(2^j t) (\phi(t) \sin(2^l t))' dt \right| \le c \sum_{j=1}^{\infty} 2^{-j} \sum_{l=1, l \neq j}^{N} 2^{-l}$$

which is bounded independent of N. Next

$$\left| \int \int \phi(t) 2^{-j/p - j/q} \sin(2^j t) \cos(2^j t) \phi'(t)) dt \right| \le c_1 2^{-j}$$

and

$$\left|\int\int \phi^2(t)2^{-j/p-j/q+j}\cos^2(2^j t)dt\right| = \left|\int\int \phi^2(t)\frac{1}{2}(1+\cos(2^{j+1}t))dt\right|$$
$$=\frac{1}{2}\int \phi^2(t)dt + c_2^{-j}.$$

We expand (3.9). Only the diagonal terms contribute. This completes the proof. □

3.6. **Step functions are dense.**

Lemma 3.15. *For all $v \in V^p$ and all partitions τ we have (recall Definition 3.6)*

$$\|v_\tau\|_{V^p} \le \|v\|_{V^p}. \tag{3.10}$$

and for all $u \in U^p$

$$\|u_\tau\|_{U^p(I)} \le \|u\|_{U^p(I)}. \tag{3.11}$$

For $v \in V^p$ and $\varepsilon > 0$ there is a partition τ so that

$$\|v - v_\tau\|_{V^p} < \varepsilon. \tag{3.12}$$

Given $u \in U^p$ and $\varepsilon > 0$ there exists τ with

$$\|u - u_\tau\|_{U^p} < \varepsilon. \tag{3.13}$$

In particular the step functions $\mathcal{S}$ are dense in V^p and $\mathcal{S}_{rc}$ is dense in U^p.

Proof. When we take the supremum over partitions for v_τ we may restrict to subsets of τ and the first statement becomes obvious. For U^p it suffices to check p atoms a,

$$\|a_\tau\|_{U^p} \le 1.$$

Density of step functions in U^p follows from the atomic definition of the space: Let $u \in U^p$ and $\varepsilon > 0$. By definition there exists a finite sum of atoms (which is a right continuous step function u_{step}) such that

$$\|u - u_{step}\|_{U^p} < \varepsilon/2$$

Let τ be the partition associated to u_{step}. Then

$$\begin{aligned}\|u - u_\tau\|_{U^p} \le & \|u_{step} - u_\tau\|_{U^p} + \|u - u_{step}\|_{U^p} \\ < & \|(u_{step} - u)_\tau\|_{U^p} + \varepsilon/2 \\ < & \varepsilon.\end{aligned}$$

which is the claim for U^p. Let $\tilde V^p$ be the closure of the step functions in V^p. Suppose there exists $v \in V^p$ with distance > 1 to $\tilde V^p$, and $\|v\|_{V^p} < 1+\varepsilon$. Such a function exists when $\tilde V^p$ is not V^p. Let $D \subset U^q$ be the subset such $B(u,v) = 0$ whenever $u \in D$ and $v \in \tilde V^p$. Since the dual space of D, $D^* = V^p/\tilde V^p$, and since v defines an element in D^* of norm > 1 there exists $u \in D$ with $B(u,v) = 1$, and a partition τ so that $\|u - u_\tau\|_{U^p} < \varepsilon$. However

$$0 = B(u, v_\tau) = B(u_\tau, v) = B(u,v) + B(u_\tau - u, v) \ge 1 - \varepsilon(1+\varepsilon)$$

which is a contradiction if $\varepsilon < \frac{1}{2}$. Hence the step functions are dense in V^p and, given $v \in V^p$ and $\varepsilon > 0$ there is a step function v_{step} with $\|v - v_{step}\|_{V^p} < \varepsilon$ and partition τ. Then

$$\begin{aligned}\|v - v_\tau\|_{V^p} \leq & \|v_{step} - v_\tau\|_{V^p} + \|v - v_{step}\|_{V^p} \\ < & \|(v_{step} - v)_\tau\|_{V^p} + \varepsilon/2 \\ < & \varepsilon.\end{aligned}$$

which is the density assertion. □

3.7. **Convolution and regularization.** Convolution by an L^1 function defines a bounded operator on U^p and V^p Ruled functions are in L^∞ and hence the product of a function in U^p or V^p with an L^1 function can be integrated.

Lemma 3.16. *Let $a = -\infty$ and $b = \infty$, $v \in V^p$ and $\phi \in L^1$. Then*

$$\|v * \phi\|_{V^p(X)} \leq \|\phi\|_{L^1} \|v\|_{V^p(X)}$$

and

$$\|u * \phi\|_{U^p(X)} \leq \|\phi\|_{L^1} \|u\|_{U^p(X)}.$$

Proof. Let τ be a partition. It suffices to consider ϕ non negative and with integral 1. Then, by convexity and Jensens inequality

$$\sum |\phi * v(t_{i+1}) - \phi * v(t_i)|^p \leq \int |\phi(h)| \sum_i |v(t_{i+1} + h) - v(t_i + h)|^p dh \leq \|v\|_{V^p}$$

The statement for U^p follows by duality: We have

$$B(\phi * a, v) = B(a, \tilde{\phi} * v)$$

with $\tilde{\phi}(t) = \phi(-t)$. □

The first part of the next result it due to Hardy and Littlewood [11]. The Besov spaces of the lemma will be explained in the proof. We include third statement for completeness, but it will not be used lateron.

Lemma 3.17. *Let $h > 0$ and $f \in V^p$. Then*

$$\|v(.+h) - v(.)\|_{L^p} \leq (2h)^{1/p} \|v\|_{V^p}. \tag{3.14}$$

In particular, if $1 < p < \infty$,

$$\|v\|_{\dot{B}^{1/p,p}_\infty} \leq c\|v\|_{V^p}$$

and

$$\|u\|_{U^p} \leq c\|u\|_{\dot{B}^1_{p,1}}$$

Proof. Let $I_j = [jh, (j+1)h]$ where

$$|v(t+h) - v(t)| \leq \max\{\sup_{[jh,(j+1)h]} v - \inf_{[(j+1)h,(j+2)h]} v, \sup_{[(j+1)h,(j+2)h]} v - \inf_{[jh,(j+1)h]} v\}.$$

For $\varepsilon > 0$ there exist two points $t_{j,0} \in I_j$ and $t_{j,1} \in I_{j+1}$ with

$$\sup_{t \in I_j} |v(t+h) - v(t)| \leq (1 - \varepsilon)|v(t_{j+1}) - v(t_j)|.$$

For simplicity we assume that v is continuous, in which case we may choose $\varepsilon = 0$, which is the only use we will make of the continuity assumption. Hence

$$\begin{aligned}\int |v(t+h) - v(t)|^p dx \le & h\Big(\sum_i |v(t_{2i+1,1}) - v(t_{2i+1,0})|^p + \sum_i |v(t_{2i,1}) - v(t_{2i,0})|^p\Big) \\ \le & 2h\|v\|_{V^p}^p.\end{aligned}$$

All partial sums on the right hand side are bounded by $2h\|v\|^p_{V^p}$ and hence the same is true for the sum. There are many equivalent norms on the homogeneous Besov space, one of them being

$$\|v\|_{\dot B^{1/p}_{p,\infty}} = \sup_{h>0} h^{-1/p}\|v(.+h) - v\|_{L^p}$$

and the bound follows from the estimate for the difference. The last statement follows by duality: The bilinear map

$$\dot B^{\frac1p}_{p,\infty} \times \dot B^{1-\frac1p}_{\frac{p}{p-1},1} \ni (f,g) \to \int f dg$$

defines an isomorphism $\dot B^{\frac1p}_{p,\infty} \to (\dot B^{1-\frac1p}_{\frac{p}{p-1},1})^*$. Here for $0 < s < 1$ and $1 \le q < \infty$

$$\|v\|_{\dot B^s_{p,q}} = \int_0^\infty (h^{-1}\|v(.+h) - v\|_{L^p})^q \frac{dh}{h}.$$

See Triebel [30] for the theory of these spaces. □

Let $\phi \in C_0^\infty$ with $\int \phi = 0$. Then it is an immediate consequence that

$$\begin{aligned}\|v*\phi\|_{L^p} =& \|(v(t+h) - v(t))\phi(h)dt\|_{L^p} \\ \le& \sup h^{-1/p}\|v(t+h) - v(t)\|_{L^p}\int h^{1/p}|\phi(h)|dh \\ \le& c\|v\|_{V^p}\end{aligned} \tag{3.15}$$

and, by duality, for $\phi \in C_0^\infty$,

$$\begin{aligned}\|u*\phi\|_{U^p} \le & \sup_{\|v\|_{V^q}\le 1} B(\phi * u, v) \\ =& \sup_{\|v\|_{V^q}\le 1} \int \phi' * uvdt \\ =& \sup_{\|v\|_{V^q}\le 1} \int u\tilde\phi' vdt \\ \le & \sup_{\|v\|_{V^q}\le 1} \|u\|_{L^p}\|\phi' v\|_{L^q} \\ \le & C\|u\|_{L^p}\end{aligned} \tag{3.16}$$

Clearly $C_0^\infty \subset V^1_{rc}$. Let $\tilde V^p \subset V^p$ be the closed subspace of functions with $f(t) = \frac12(\lim_{h\to 0}(f(t+h) + f(t-h)))$. We consider functions on $\mathbb{R}$. If $v \in V^p$ is continuous then

$$B(\phi_h * a, v) \to B(a,v) \text{ as } h \to 0$$

for all atoms a. Here $\phi \in L^1$ with $\int \phi dx = 1$ and $\phi_h(x) = h^{-1}\phi(x/h)$. If moreover ϕ is symmetric then

$$\phi_h * v \to v$$

pointwise for all $v \in \tilde V^p$ and $B(\phi_h * u, v) = B(u, \phi_h * v)$ for all $u \in U^q$ and $v \in V^p$.

Lemma 3.18. *We have*
$$B(\phi_h * u, v) \to B(u,v)$$
for $u \in U^p(\mathbb{R})$ *and* $v \in V^q \cap C$ *and*
$$\phi_h * v \to v$$
in the weak * *topology for* $v \in \tilde V^p(\mathbb{R})$ *for* $1 \le p < \infty$.

Proof. Only the last statement needs a proof. By definition and the pointwise convergence $B(u, \phi_h * v) \to B(u,v)$ for all $u \in \mathcal{R}_{rc}$. This implies weak star convergence. □

3.8. **Density of test functions and duality.**

Lemma 3.19. *Test functions* C_0^∞ *are weak* dense in* V^p.

Proof. Step functions are dense in V^p, and it suffices to verify that step functions can be approximated by C_0^∞ functions in the weak$*$ sense. Moreover it suffices to consider test functions with a partition consisting of a single point, which we choose to be 0. Hence we reduce the problem to a proof for three functions. We fix $\phi \in C_0^\infty(\mathbb{R})$, identically 1 in $[-1,1]$, and $\eta \in C^\infty(\mathbb{R})$ supported in $(0,\infty)$ and identically 1 for $t \ge 1$. Then for $u \in \mathcal{S}_{rc}$ checking the definition shows
$$B(u, \phi(t/j)) \to B(u,1)$$
and with $v(t) = 0$ for $t \neq 0$ and $v(0) = 1$
$$B(u, \phi(jt) \to B(u,v)$$
and, with $v(t) = 0$ for $t \le 0$ and 1 for $t > 0$
$$B(u, \phi(t/j)\eta(jt)) \to B(u,v)$$
with $j \to \infty$. □

Corollary 3.20. *We have*
$$\|u\|_{U^p(X)} = \sup\{B(u,v) : v \in C_0^\infty, \|v\|_{V^q(X^*)} = 1\}.$$

Lemma 3.21. *The bilinear map* B *defines a surjective isometry*
$$\tilde V^p(X^*) \to (U^q \cap C(X))^*, \frac{1}{p} + \frac{1}{q} = 1, 1 < p, q < \infty.$$

Proof. The kernel of the duality map composed composed with the inclusion $(U^p \cap C) \subset U^p$ consists exactly of those elements of V^q which are nonzero at most at countably many points. We claim that the duality map is an isometry. Let $v \in \tilde V^p$, and let a be an atom so that
$$\|v\|_{V^p} \le (1+\varepsilon) B(a,v)$$
If ϕ_h is a symmetric mollifier then, if h is sufficiently small
$$B(a, \phi_h * v) = B(\phi_h * a, v)$$
which shows that the dualtiy map is an isometry.

It remains to prove surjectivity. Let $L : U^p \cap C(X) \to \mathbb{R}$ be linear and continuous. By the theorem of Hahn-Banach there is a extension with the same norm to U^p, and by duality there is $v \in V^q$ with $\|v\|_{V^q} = \|L\|$ and $L(u) = B(u,v)$ for all $u \in U^p$. Changing v at a countable set does not change the image in $(U^p \cap C(X))^*$, hence we may choose $v \in \tilde V^p$. □

In the sequel we identify $u(a)$ resp. $u(b)$ with the limit from the right resp. the left.

Lemma 3.22. *Let $u \in U^q$ and $v \in U^p$, $1/p + 1/q = 1$ and let $(t_j)_{j\to 1}$ be the points where both v and u have jumps, and denote the size of the jumps by $\Delta u(t_j)$. Then*

$$B(u,v) + B(v,u) = \sum_j \Delta u(t_j)\Delta v(t_j) + u(b)v(b) \tag{3.17}$$

Proof. The right hand side of (3.17) is continuous with respect to $u \in V^q$ and $v \in V^p$, with the jump understood as the difference between the limit from the right and the left - the sum over the jumps to the power p is bounded by the V^p norm. The left hand side is continuous with respect to $u \in U^q$ and $v \in V^p$, and it suffices to verify the formula for $u, v \in \mathcal{S}_{rc}$ with joint partition (where we add $t_0 = a$) $a = t_0 < t_1 \dots t_N < b$. Then

$$B(u,v) + B(v,u) = \sum_{j=1}^N u(t_j)(v(t_j) - v(t_{j-1}) + (u(t_j) - u(t_{j-1})v(t_j)$$

$$= \sum_{j=1}^N u(t_j)(v(t_j) - v(t_{j-1})) - u(t_{j-1})(v(t_j) - v(t_{j-1}) + u(b)v(b) - u(t_0)v(t_0).$$

□

We define

$$V_C^q = \{v \in V^q \cap C, v(b) = 0\}. \tag{3.18}$$

Lemma 3.23. *The map*

$$U^p(X^*) \to (V_C^q)^*,$$
$$u \to (v \to B(u,v))$$

is a surjective isometry.

Proof. By the duality estimates the duality map is defined, and it is an isometry since the space V_C^q is weak star dense in V^q. Let $L : V_C^q \to \mathbb{R}$ by linear and continuous. By Hahn Banach L can be extended to continuous linear form on $\tilde{L} \in (V^q)^*$. Since $U^q \subset V_{rc}^q$ by an abuse of notation $L \in (U^q)^*$ and there exists $\tilde{u} \in V^p$ such that

$$B(w, -\tilde{u}) = \tilde{L}(w)$$

for all $w \in U^q$. We define (with $t\pm$ the limit from the left resp. the right)

$$u(t) = \tilde{u}(t+) - \tilde{u}(a).$$

Then $u \in \cap_{\tilde{p}>p} U^{\tilde{p}}$ and by Lemma 3.22 below, for all $v \in V_C^p$,

$$\begin{aligned}
L(v) =& L(v - v(a)) + v(a)L(1) \\
=& B(v - v(a), \tilde{u}) + v(a) \lim_{t\to a} L(\chi_{(t,b)}) \\
=& B(v - v(a), u) - (v(b-) - v(a))\tilde{u}(a) + v(a)\tilde{u}(a) \\
=& B(u, v - v(a)) + (\tilde{u}(b) - \tilde{u}(a))v(a) \\
=& B(u,v)
\end{aligned}$$

where we used that $v(b) = 0$ and that v is continuous.

For every partition we have $u_\tau \in U^p$, with

$$\|u_\tau\|_{U^p} \le \sup_{v\in V^p_C, \|v\|_{V^p}\le 1} B(v,u_\tau) = \sup_{\|v_\tau\|_{V^p}=1} L(v_\tau)$$

Since $u \in V^p_{rc}$ there is a sequence of partitions τ_i so that $u_{\tau_i} \to u \in V^p$ and hence the sequence converges uniformly. Thus for every step function v

$$B(u_{\tau_i}, v) \to B(u,v).$$

Since step functions are dense in V^q even

$$B(u_{\tau_i}, v) = B(u, v_{\tau_i}) \to B(u,v)$$

For all $v \in V^q$. Let U^{p**} be the bidual space of U^p , which we consider as isometric closed subspace of X^{**}. By an abuse of notation we consider u as element of U^{p**}. Then

$$B(u_{\tau_i}, v) \to u(v)$$

for all $v \in V^q$ and the distance between u and U^p in U^{p**} is zero, and hence $u \in U^p$. □

3.9. **Consequences of Minkowski's inequality.** For a Banach space Y we denote by $L^p(Y)$ the weakly measurable maps with values in Y; for which the norm is p integrable.

Lemma 3.24. *We have for* $1 < p \le q < \infty$

$$\|u\|_{L^q_x(U^p)} \le \|u\|_{U^p(L^q_x)} \tag{3.19}$$

and

$$\|v\|_{V^p(L^q_x)} \le \|v\|_{L^q_x(V^p)}. \tag{3.20}$$

Proof. It suffices to verify the first inequality for a p atom

$$a(t,x) = \sum \chi_{[t_i,t_{i+1})}(t)\Phi_i(x)$$

with values in L^q. This is a function of x and t. Then $t \to a_x(t)$ is a step function. Let

$$f(x) = \left(\sum_i |\Phi_i(x)|^p\right)^{1/p}$$

Then

$$\begin{aligned}\|a\|_{L^q_x(U^p)} &= \left(\int f(x)^q dx\right)^{1/q}\\ &= \left(\int \left(\sum_j |\Phi_j(x)|^p\right)^{q/p}\right)^{1/q}\\ &\le \left(\sum_j \|\Phi_j\|^p_{L^q}\right)^{1/p}\\ &\le 1\end{aligned}$$

where we use Minkowski's inequality for the first inequality. The argument for the V^p space is similar. □

The argument works the same way if we consider Banach space valued functions in U^pL^q etc.

3.10. The bilinear form as integral.

Here we consider scalar valued functions.

Definition 3.25. *Let $v \in V^p(a,c)$ and $u \in U^q(a,c)$. We define for $a \le s < t \le b$*

$$\int_s^t v du := B_{(s,t)}(u - u(s), v) + (u(t) - u(t-))v(t) \tag{3.21}$$

and

$$\begin{aligned}\int_s^t u dv := -\int_s^t v du + \sum_j (u(t_j) - u(t_j-))(v(t_j) - v(t_j-)) \\ + u(t-)v(t-) - u(s)v(s+) + u(t)(v(t+) - v(t-)) + v(t)(u(t) - u(t-))\end{aligned} \tag{3.22}$$

with the sum over all joined jumps in (s,t).

The second definition is partly motivated by

(1) The integration by parts formula (3.17). It should reduce to integration by parts if $v \in U^q$, and if there are no jumps at t
(2) The desire to have a certain symmetry with time reversion if v is continuous the left and u is continuous from the right.
(3) We want the integral to be additive in the interval.

Lemma 3.26. *For $u \in U^q$ and $v \in V^p$, $1/p + 1/q = 1$ we have*

$$\int_a^c v du = \int_a^b v du + \int_b^c v du$$

and

$$\int_a^c u dv = \int_a^b u dv + \int_b^c u dv.$$

With the obvious notation,

$$\Big\| \int_a^t u dv \Big\|_{V^p} \le \|u\|_{U^q} \|v\|_{V^p} \tag{3.23}$$

and

$$\Big\| \int_a^t v du \Big\|_{U^q} \le \|u\|_{U^q} \|v\|_{V^p}. \tag{3.24}$$

Proof. It suffices to check the first formula for atoms u. Suppose that $t_j < b \le t_{j+1}$. On both sides we have a sum over

$$v(t_{j+1})(u(t_{j+1}) - u(t_j)).$$

For the second formula we see from the definition

$$\int_a^c u dv = \int_a^b u dv + \int_b^c u dv$$

where we have to check the contribution at $t = b$.

Formally, for smooth functions

$$\begin{aligned}
B(\int_a^t vdu, w) = \int_a^b & w(t)v(t)u'(t)dt \\
=&B(u, vw) \\
\le&\|vw\|_{V^q}\|u\|_{U^p} \\
\le&2|v\|_{V^q}\|w\|_{V^q}\|u\|_{U^p}
\end{aligned} \tag{3.25}$$

which formally implies (3.24).

For a rigorous proof we verify the formula in the case when u is a atom, and v and w are step function with a common partition all functions. Then $\int_a^t vdu$ is a right continuous step function and

$$\sum_j (v(t_j)(u(t_j) - u(t_{j-1}))w(t_j) = \sum_j [v(t_{j+1})w(t_{j+1}) - v(t_j)w(t_j)]u(t_j)$$

where we neglect the boundary terms. We apply Hölder's inequality to bound the expression by

$$\Big(\sum |v(t_{j+1})w(t_{j+1}) - v(t_j)w(t_j)|^q\Big)^{1/q} \Big(\sum |u(t_j)|^p\Big)^{1/p}.$$

Again formally for smooth functions

$$\begin{aligned}
B(w, \int_a^t udv) =& -\int_a^b vwu'dt + (w(b) - w(a))\int_a^b uv'dt \\
=& \int_a^b v(uw)'dt - (w(b) - w(a))\int_a^b vu'dt \\
& - u(b)v(b)w(b) + u(a)v(a)w(a) \\
& + (w(b) - w(a))(u(b)v(b) - u(a)v(a)) \\
=& B(uw, v) - (w(b) - w(a))B(v, u)
\end{aligned} \tag{3.26}$$

if $u(a) = w(a) = 0$. This implies formally (3.24). For a rigorous proof we apply integration by parts several times. First

$$\begin{aligned}
\int_{t-}^{t+} udv =&(u(t) - u(t-))(v(t) - v(t-)) + u(t)v(t+) - u(t-)v(t-) \\
& - v(t)(u(t) - u(t-)) \\
=&u(t)(v(t+) - v(t-))
\end{aligned}$$

and

$$\int_t^{t+} udv = u(t)(v(t+) - v(t))$$

and hence l^p sum over the jumps is bounded. Thus the bounded reduces to the bound for

$$B(w, \int_a^t vdu)$$

and by the same token to

$$B(\int_a^t vdu, w)$$

which we have proven above. ☐

Sometimes it is convenient to have a notation for spaces of derivatives of functions in U^p resp. V^p.

Definition 3.27. *We define dU^p as the space of all distributions f with $\int_{-\infty}^x f \in U^p$, equipped with the norm in U^p. Similarly, let dV^p be the space of all distributions which have an antiderivative in $\tilde V^p_{rc}$, equipped with the obvious norm.*

In particular we require the integrals to be in L^1_{loc} and that all one sided limits exist.

3.11. **Differential equations with rough paths.** This type of study was initiated by Lyons [21]. We will only scratch on the surface. We observe that the duality mapping extends the Young integral.

We consider the differential equation

$$\dot y = F(y,x)\dot x, \qquad y(0) = y_0$$

where $x \in U^2$ and F is a bounded Lipschitz function continuously Frechet differentiable with respect to Y, and $d_Y F$ is uniformly Lipschitz continuous. We denote the bound for F by C_0, the Lipschitz bound by L_0, the bound for $d_Y F$ by C_1, and the Lipschitz bound for $d_Y F$ by L_1.

Suppose that y is a solution, i.e

$$y(t) = y(a) + \int_a^t F(y,x)dx$$

Then, by (3.24)

$$\begin{aligned} \|y(t)-y(a)\|_{U^2} \le & \|F(y,x)\|_{V^2}\|x\|_{U^2} \\ \le & (C_0 + L_0(\|y\|_{V^2}) + \|x\|_{V^2})\|x\|_{U^2} \end{aligned} \tag{3.27}$$

It is trivial that there is a unique solution if x is a step function in $\mathcal{S}_{rc}$ - for that we consider a finite number of differences. We shall construct a solution to the initial value problem for $\|x\|_{U^p}$ small. This implies existence of a unique solution since we may first approximate x by a step function, and then solve the differential equation on each of the intervals of the step function.

We want to construct a solution as fixed point of

$$y(t) = y_0 + \int_0^t F(y(s),x(s))\dot x ds.$$

We claim that there is a unique solution in U^2 provided

$$\|x\|_{U^2} < \varepsilon$$

with ε sufficiently small. Let

$$y(t) = y_0 + \int_0^t F(\tilde y(s),x(s))\dot x ds.$$

Now, by (3.27),

$$\|y - y(a)\|_{U^2} \le (C_0 + \varepsilon L_0 + L_0(\|\tilde y\|_{V^2}))\|x\|_{U^2}$$

and we obtain a uniform bound on the iteration provided $L_0\|x\|_{U^2} \le \frac{1}{2}$. If $\tilde y_1, \tilde y_2 \in U^2$ and y_i is defined by the Young integral above we get

$$\begin{aligned}\|y_2 - y_1\|_{U^2} \le & 2\|F(\tilde y_2, x) - F(\tilde y_1, x)\|_{V^2}\|x\|_{U^2} \\ \le & 2\|\frac{F(\tilde y_2, x) - F(\tilde y_1, x)}{\tilde y_2 - \tilde y_1}(\tilde y_2 - \tilde y_1)\|_{V^2}\|x\|_{U^2} \\ \le & 2\Big(\|\frac{F(\tilde y_2, x) - F(\tilde y_1, x)}{\tilde y_2 - \tilde y_1}\|_{V^2}\|\tilde y_1 - \tilde y_1\|_{sup} \\ & + \|\frac{F(\tilde y_2, x) - F(\tilde y_1, x)}{\tilde y_2 - \tilde y_1}\|_{sup}\|\tilde y_1 - \tilde y_1\|_{V^2}\Big)\|x\|_{U^2} \\ \le & 2(L_1(1 + \|\tilde y_2 - \tilde y_1\|_{V^2} + \|x\|_{V^2})\|\tilde y_2 - \tilde y_1\|_{V^2}\|x\|_{U^2}.\end{aligned}$$

We easily construct a unique solution by a standard contraction argument. The modifications for U^p $p < 2$ are as follows. The differentiability requirements on F are weaker: Let $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. The apriori estimate requires few changes and we concentrate on the contraction, for which we consider (with $L_{p/q}$ the p/q Hölder exponent of the derivative of F)

$$\begin{aligned}\|F(\tilde y_2, x) - F(\tilde y_1, x)\|_{V^q} = & \sup_\tau \Big(\sum \big|(F(\tilde y_2(t_{i+1}), x(t_{i+1})) - F(\tilde y_2(t_i), x(t_i))) \\ & - (F(\tilde y_1(t_{i+1}), x(t_{i+1})) - F(\tilde y_1(t_i), x(t_i)))\big|^q\Big)^{1/q} \\ \le & \sup_\tau \Big[\Big(\sum_i |F(\tilde y_2(t_{i+1}), x(t_i)) - F(\tilde y_2(t_i), x(t_i)) \\ & + F(\tilde y_1(t_{i+1}), x(t_i)) - F(\tilde y_1(t_i), x(t_i))|^q\Big)^{1/q} \\ & + \Big(\sum_i |F(\tilde y_2(t_{i+1}), x(t_{i+1})) - F(\tilde y_2(t_{i+1}), x(t_i)) \\ & - (F(\tilde y_1(t_{i+1}), x(t_{i+1}))) - F(\tilde y_1(t_{i+1}), x(t_i))|^q\Big)^{1/q}\Big] \\ \le & \sup_\tau \Big(L_1^q\|\tilde y_2 - y_1\|_{sup}^q(\|\tilde y_2 - \tilde y_1\|_{V^p}^p + \|x\|_{V^p}^p)^{1/q}\|x\|_{sup}\Big)\end{aligned}$$

We obtain the contraction as above.

Theorem 3.28. *Let $1 < p \le 2$, $F : X \times Y \to Y$ be bounded, uniformly Lipschitz continuous, Frechet differentiable with respect to X and Y, and dF is Hölder continuous with respect to y with Hölder exponent $p - 1$. We study*

$$dy = F(x, y)dx, y(a) = y_0$$

Then there exists a unique solution $y \in U^p(Y)$ if $x \in U^p$ if $1 \le p \le 2$ and $y \in V^p$ if $x \in V^p$ and dF is Hölder continuous with exponent $s > p - 1$.

3.12. **The Brownian motion.** The Brownian motion is almost surely in V^p for $p > 2$. We denote by $B_t(\omega)$ the path of the Brownian motion as a function of t and the element of the probability space ω. If the Brownian motion would be in U^2 with positive probability we could solve stochastic differential equations in a pointwise sense. The 2-variation however is almost certainly infinite.

The regularity of the Brownian motion is characterized by the following fairly sharp result of Taylor [29] , see also [7].

Theorem 3.29. *Let*

$$\psi_{2,1}(h) = \begin{cases} h^2 & \text{for } h \geq e^{-e} \\ \frac{h^2}{\ln\ln(1/h)} & \text{if } h < e^{-e} \end{cases}$$

There exists $\eta > 0$ *so that*

$$\mathbb{E}(\exp(\frac{\eta}{T}\|B\|^2_{\psi_{2,1};[0,T]}) < \infty$$

where

$$\|B\|_{\psi_{2,1};[0,T]} = \inf\{M > 0 : \sup_{\tau} \sum \psi_{2,1}(|B_{t_{i+1}} - B_{t_i}|/M) \leq 1\}.$$

Moreover, if

$$\frac{h^2}{\psi(h)\ln\ln(1/h)} \to 0 \text{ as } h \to 0$$

then

$$\sup_{\tau_T} \sum \psi(|B_{t_{i+1}} - B_{t_i}|) = \infty.$$

See Theorem 13.15 and Theorem 13.69 in [7]. This result deviates from the V^p spaces by an iterated logarithm.

Let (Ω, μ) be a probability space with a filtration μ_t, $t \in \mathbb{R}$, $f \in L^p$ and $f_t = \mathbb{E}(f, \mu_t)$. Then

$$\|f_t\|_{L^p(\Omega, V^2_w)} \leq c_p \|f\|_{L^p} \tag{3.28}$$

is a consequence of Doob's oscillation lemma for martingals [23], see also Bourgain's proof of p-variation estimate [2]. A weaker version is due to Lepingle [19].

For the Brownian motion B_t we obtain

Theorem 3.30.

$$\|B_t\|_{L^p(\Omega, V^2_w([0,1)))} \leq c_p.$$

This has been a motivation to introduce V^p_w.

3.13. **Adapted function spaces.** We want to bound the norm in U^p or V^p by integrating against smooth functions. This is the contents of the next lemma.

Lemma 3.31. *Suppose that* T *is a distribution so that*

$$\sup\{T(v_t) : v \in C_0^\infty, \|v\|_{V^q} \leq 1\} = C_1 < \infty$$

then there exists a unique $u \in U^p$ *with*

$$T(v_t) = -B(u, v)$$

and

$$\|u\|_{U^p} = C_1.$$

Suppose that T *is a distribution so that*

$$\sup\{T(u_t) : u \in C_0^\infty, \|u\|_{U^q} \leq 1\} = C_2 < \infty$$

then there exists a unique $V \in V^p_{rc}$ *with*

$$T(u_t) = B(u, v)$$

and

$$\|u\|_{U^p} = C_2.$$

Proof. Let η_j be a Dirac sequence and

$$u_j = T * \eta_j \in C^\infty$$

Then $u_j - u_j(a) \in U^p(a,b)$ for all a and b and hence by duality

$$\|u_j - u_j(a)\|_{U^p(a,b)} \leq C_1$$

In particular for $t > a$

$$|u_j(t) - u_j(a)| \leq C_1$$

and since a and b are arbitrary

$$\|u_j(t) - u_j(a)\|_{sup} \leq C_1$$

and we may assume that

$$T(v_t) = \int \tilde{u} v_t dt$$

for a measurable function $\tilde{u}$ which satisfying

$$\|\tilde{u}\|_{sup} \leq C_1$$

It defines a bounded linear form on V^q. We choose v to be suitable atoms, and obtain uniform bounds of u_j in V^p, and hence, after modification on a set of measure 0 also $\tilde{u} \in V^p$. As for the duality argument we see that $\tilde{u} \in V^p_{rc}$, and, as there

$$\tilde{u} - \tilde{u}(a) \in U^p((a,b)$$

for all a and b.

As for V^p one sees that C_0^∞ is weak* dense in U^p. So

$$\|v - v(b)\|_{V^p} = \sup\{\int v u_t dt : u \in C_0^\infty, \|u\|_{U^q} = 1\}.$$

A time reversal and a repetition of the arguments above implies the full assertion. □

we observe that there are not more than obvious changes if we consider Hilbert spaces valued functions, and if we replace the product by the inner product.

We briefly survey constructions going back to Bourgain, which have become standard. The following situation will be of particular interest. Let $t \to S(t)$ be a continuous unitary group on a Hilbert space H. We define U^p_S and V^p_S by

$$\|u\|_{U^p_S} = \|S(-t)u(t)\|_{U^p(H)}.$$

By Stone's theorem unitary groups are in one-one correspondence with selfadjoint operators, in the sense that

$$i\partial_t u = AU$$

with a self adjoint operator defines a unitary group $S(t)$ and vice versa. At least formally

$$i\partial_t(S(-t)u(t)) = S(-t)(i\partial_t u - Au)$$

and hence the duality assertion is

$$\|u\|_{U^q_S} = \sup_{\|v\|_{V^p_S} \leq 1} B(S(-t)u(t), S(-t)v(t)).$$

Now suppose that - again formally -

$$i\partial_t u + Au = f$$

then, if we choose by Duhamel's formula the solution

$$u(t) = \int_{-\infty}^{t} S(t-s)f(s)ds.$$

A related construction goes back to Bourgain. He defines

$$\|u\|_{X_S^{0,b}} = \|S(-t)u(t)\|_{H^b L^2} \tag{3.29}$$

where the Sobolev space H^b is defined by the Fourier transform,

$$\|f\|_{H^b} = \|(1+|\tau|^2)^{b/2}\hat{f}\|_{L^2}$$

Clearly

$$X_S^{0,b} \subset X_S^{0,b'}$$

whenever $b \geq b'$. We may use a Besov refinement of the right hand side of (3.29), i.e.

$$\|u\|_{\dot{X}^{s,b,q}} = \left(\sum_{N \in 2^{\mathbb{Z}}} N^{sq} \|u_N\|_{H^b(L^2)}^q \right)^{1/q}$$

where we choose a disjoint partition $A_j = \{(\tau,\xi) : 2^N \leq |\tau + \phi(\xi)| \leq 2^{1+N}\}$ and define u_N by the Fourier multiplication by the characteristic function of A_N.

Then

$$\dot{X}_S^{0,\frac{1}{2},1} \subset U_S^2 \subset V_S^2 \subset \dot{X}^{0,\frac{1}{2},\infty}$$

follows from Lemma 3.17.

There is an obvious generalization to the case of time dependent operators $A(t)$. Definitions are simple, but this often leads to technical questions.

Now

$$\mathcal{F}_{t,x}(S(-t)u)(\tau,\xi) = \mathcal{F}_t e^{-it\phi(\xi)}\hat{u}(t,\xi) = \mathcal{F}_{t,x}u(\tau - t\phi(\xi),\xi)$$

and hence by the formula of Plancherel and a translation in τ variable

$$\|u\|_{X^{0,b}} = \|(1+\tau^2)^{b/2}\mathcal{F}_{t,x}(u)(\tau - t\phi(\xi),\xi)\|_{L^2} = \|(1+(\tau+\phi(\xi))^2)^{b/2}\mathcal{F}_{t,x}(u)\|_{L^2}.$$

3.13.1. *Strichartz estimates.* We want to use this construction for dispersive equations. There A is often defined by a Fourier multiplier, most often even by a partial differential operator with constant coefficients.

We consider the Schrödinger equation

$$i\partial_t u + \Delta u = 0 \qquad \text{in } [0,\infty)$$

$$u(0) = u_0 \qquad \text{on } \mathbb{R}^d$$

Let $u(t) = 0$ for $t < 0$ and the solution otherwise. Then

$$\|u\|_{U_S^1} = \|u_0\|_{L^2(\mathbb{R}^d)}.$$

One of the Strichartz estimates states

$$\|u\|_{L_t^p L_x^q} \leq \|u_0\|_{L^2} \tag{3.30}$$

whenever

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \qquad 2 \leq p,q, \quad (p,q,d) \neq (2,\infty,2).$$

We claim that this implies

$$\|u\|_{L_t^p L_x^q} \leq c\|u\|_{U^p}.$$

It suffices to verify this if $S(-t)u$ is an atom with partition $(t_1, t_2 \dots t_n)$. Then, with $t_{n+1} = \infty$, by the Strichartz estimate

$$\|u\|_{L^p_t(t_j,t_{j+1});L^q_x)} \le c\|u(t_j)\|_{L^2}.$$

We raise this to the pth power, and add over j. Then

$$\|u\|_{L^pL^q} \le c\Big(\sum \|u(t_j)\|_{L^2}^p\Big)^{1/p} \le c$$

since $S(-t)u$ is a p atom.

Consider $v(t) = \int_{-\infty}^t S(t-s)f(s)ds$ and let $\tau = (t_j)$ be a partition. Then

$$v(t_j) - S(t_j - t_{j-1})v(t_{j-1}) = \int_{t_{j-1}}^{t_j} S(t_j - t)f(t)dt$$

and by the Strichartz estimate

$$\|S(-t_j)v(t_j) - S(-t_{j-1})v(t_{j-1})\|_{L^2} \le c\|f\|_{L^{p'}_t L^{q'}_x}$$

and

$$t \to S(-t)v(t)$$

is continuous.

We take the power p' and sum over j to reach the conclusion

$$\|v\|_{V^{p'}_S} \le c\|f\|_{L^{p'}L^{q'}}$$

This implies the dual estimate to (3.30). If $p > 2$ we may combine the estimates with an embedding to obtain the full Strichartz estimate. In particular we arrive at the non symmetric improvement for the Strichartz estimate

$$\|u\|_{L^\infty(L^2)} + \|u\|_{L^{q_0,p_0}} \le c\Big(\|u_0\|_{L^2} + \|f\|_{L^{q_1'p_1'}}\Big)$$

if both (q_1, p_1) and (q_0, p_0) are Strichartz pairs, but not necessarely the same ones.

We prove this estimate over the interval $(0, \infty)$ and extend u by 0 to negative t. Then

$$\|u\|_{L^\infty(L^2)} + \|u\|_{L^{p_0,q_0}} \le c\|u\|_{U^{p_0}} \le c\|u\|_{V^{p_1'}} \le c\|u_0\|_{L^2} + \|f\|_{L^{p_1',q_1'}}.$$

Lemma 3.32. *The following estimates hold for Strichartz pairs*

$$\|u\|_{L^{p,q}} \le c\|u\|_{U^p}$$

and

$$\left\|S(t)u_0 + \int_0^t S(t-s)f(s)ds\right\|_{V^{p'}} \le c(\|u_0\|_{L^2} + \|f\|_{L^{p',q'}}).$$

3.13.2. *Estimates by duality.* We return to duality questions and calculate formally

$$\begin{aligned}
\|u\|_{U^q_S} &= \sup_{\|v\|_{V^p_S}\le 1} |B(S(-t)u(t), S(-t)v(t))| \\
&= \sup_{\|v\|_{V^p_S}\le 1} \left|\int_{\mathbb{R}} \langle \partial_t S(-t)u(t), S(-t)v(t)\rangle dt\right| \\
&= \sup_{\|v\|_{V^p_S}\le 1} |-i\langle S(-t)(i\partial_t u - Au), S(-t)v\rangle dt| \\
&= \sup_{\|v\|_{V^p_S}\le 1} \int_{\mathbb{R}} \langle f, v\rangle dt
\end{aligned} \tag{3.31}$$

with a similar statement for V^p_S. This observation will be crucial for nonlinear dispersive equations.

Lemma 3.33. *Let $\phi \in C^\infty(\mathbb{R}^d)$ be a real polynomial and let S be the unitary group defined by the Fourier multiplier $e^{it\phi(\xi)}$. Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let T be a tempered distribution in $(a,b) \times \mathbb{R}^d$ which satisfies*

$$\sup\{|T(\bar{u})| : u \in C_0^\infty((a,b) \times \mathbb{R}^d), \|u\|_{U^p_S} \leq 1\} = C_1 < \infty$$

Then there is a unique $v \in V^q_{S,rc}(a,b)$ with

$$T(u) = \int v\overline{iu_t + \phi(D)u}dxdt$$

and $\|v\|_{V^q} = C_1$. Let T be a distribution in space time which satisfies

$$\sup\{|T(\bar{v})| : v \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d), \|v\|_{V^p_S}\} = C_2 < \infty$$

Then there is a unique $u \in U^q_S$ with

$$T(\bar{v}) = \int u\overline{iv_t + \phi(D)v}dxdt$$

and $\|u\|_{U^q} = C_2$.

Proof. Fourier multiplication commutes with the evolution. We convolve T with the inverse Fourier transform of a nonnegative function with compact support. To this we apply Lemma 3.19. But this gives the full statement. □

The theorem implies existence of a weak solution to

$$i\partial_t u + \phi(D)u = f, \quad u(a) = 0,$$

together with an estimate for u.

3.13.3. *High modulation estimates.* We denote by $f(D)$ the Fourier multiplier defined by a function f. Let

$$f = 1 - \chi(\tau/\Lambda)$$

where τ is the Fourier variable corresponding to t and χ is an approximate characteristic function, i.e. χ is supported on a ball of radius 2, and identically 1 on a ball of radius 1.

Lemma 3.34. *The following estimate holds.*

$$\|f(D)v\|_{L^2} \leq c\Lambda^{-1/2}\|v\|_{V^2}$$

Suppose the group $S(t)$ is defined by the Fourier multiplier $e^{it\phi(\xi)}$ then, with

$$f(D) = 1 - \chi(\tau + \phi(\xi))$$

$$\|f(D)u\|_{L^2} \leq c\Lambda^{-1/2}\|v\|_{V^2_S}$$

Proof. We have

$$\mathcal{F}_t(e^{-it\phi(\xi)}\hat{u}(t,\xi)) = \mathcal{F}_{x,t}u(\tau - \phi(\xi), \xi)$$

and the second claim follows from the first one. Let

$$g = \mathcal{F}^{-1}\chi(\xi/\Lambda).$$

Then

$$g(t) = \Lambda^{-1}(\mathcal{F}^{-1}\chi)(\Lambda\xi)$$

and

$$\begin{aligned}\Big\|\int(v(t+h)-v(t))g(h)dh\Big\|_{L^2} & \\ \le & \sup_h |h|^{-1/2}\|v(t+h)-v(t)\|_{L^2}\int |h|^{1/2}\Lambda^{-1/2}|\mathcal{F}^{-1}\chi(h\Lambda)|dh \\ \le & c\|u\|_{V^2}\Lambda^{-1/2}\int |h|^{1/2}|\mathcal{F}^{-1}\chi|dh.\end{aligned}$$

□

4. Convolution of measures on hypersurfaces, bilinear estimates and local smoothing

We recall the transformation formula for $U, V \subset \mathbb{R}^d$, $\phi : U \to V$ a diffeomorphism,

$$\int_V f dm^d = \int_U f \circ \phi |\det D\phi| dm^d$$

and its relative, the area formula for

$$\phi : U \to S \subset \mathbb{R}^n,$$

ϕ differentiable and bijective,

$$\int_S f d\mathcal{H}^d = \int_U f \circ \phi (\det D\phi^T D\phi)^{1/2} dm^d.$$

The coarea formula deals with

$$\phi : U \to V \subset \mathbb{R}^n$$

surjective and $n \le d$. Then the coarea formula states

$$\int_V \int_{\phi^{-1}(y)} f d\mathcal{H}^{d-n} dm^n(y) = \int_U f \det(D\phi D\phi^T)^{1/2} dm^d.$$

The Fourier transform transforms a product into a convolution, and vice versa. Let Σ_1 and Σ_2 be two $d-1$ dimensional hypersurfaces in $\mathbb{R}^d$ such that for all $x_i \in \Sigma_i$ the tangent spaces are transversal.

We assume that Σ_1 and Σ_2 are nondegenerate level sets of functions ϕ_1 and ϕ_2.

Let h be a continuous function. Then, by the coarea formula

$$\int_{\mathbb{R}^d} f(x) h \circ \phi_1(x) dm^d(x) = \int_{\mathbb{R}} h(s) \int_{\phi_1^{-1}(s)} f(x) |\nabla \phi_1|^{-1}(x) d\mathcal{H}^{d-1}(x) ds.$$

This motivates the notation

$$\delta_\phi = |\nabla \phi|^{-1} d\mathcal{H}^{d-1}\Big|_{\Sigma_1}.$$

4.1. **A convolution estimate.**

Theorem 4.1. *Let $\Sigma_i \subset \mathbb{R}^d$ hypersurfaces and ϕ_i as above, and f_i square integrable functions on Σ_i. Then*

$$\|f_1 \delta_{\phi_1} * f_2 \delta_{\phi_2}\|_{L^2(\mathbb{R}^d)} \le C \|f_1 |\nabla \phi_1|^{-1/2}\|_{L^2(\Sigma_1)} \|f_2 |\nabla \phi_2|^{-1/2}\|_{L^2(\Sigma_2)}$$

where with $\Sigma(x,y) = \{y + \Gamma_1\} \cap \{x + \Gamma_2\}$

$$C = \sup_{x \in \Sigma_1, y \in \Sigma_2} C(x,y),$$

and where $C(x,y)$ is the square root of

$$\int_{\Sigma(x,y)} \left[|\nabla \phi_1(z-y)|^2 |\nabla \phi_2(z-x)|^2 - \langle \nabla \phi_1(z-x), \nabla \phi_2(z-y) \rangle^2 \right]^{-1/2} d\mathcal{H}^{d-2}.$$

Proof. Let f_i be measurable functions in a neighborhood of Σ_i, let h be countinuous and nonnegative, and $g_i = h \circ \phi_i$. Then, by Fubini,

$$\begin{aligned}
&\|f_1 g_1 * f_2 g_2\|_{L^2}^2 \\
&\quad \le \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} |f_1(x)|^2 g_1(x) g_2(z-x) dm^d(x) \int\limits_{\mathbb{R}^d} |f_2(y)|^2 g_2(y) g_1(z-y) dm^d(y) dm^d(z) \\
&\quad = \int_{\mathbb{R}^{2d}} |f_1(x)|^2 g_1(x) |f_2(y)|^2 g_2(y) \int g_2(z-x) g_1(z-y) dm^d(z) dm^{2d}(x,y).
\end{aligned}$$

By the coarea formula

$$\int g_2(z-x) g_1(z-y) dm^d = \int_{\mathbb{R}^2} h(s) h(t) I(s,t) ds dt$$

where, with

$$\Sigma_{s,t} = \{z : \phi_1(y+z) = s, \phi_2(x+z) = (t)\}$$

and

$$\rho(s,t,z) = \big| |\nabla \phi_1(z-y)|^2 |\nabla \phi_2(z-x)|^2 - (\nabla \phi_1(z-y) \cdot \nabla \phi_2(z-x))^2 \big|^{-1}$$

$$I(s,t) = \int_{\Sigma_{s,t}} \rho(s,t,z) d\mathcal{H}^{d-2}(z).$$

Here we suppress the dependence on x and y, but we set

$$\gamma(x,y) = I(0,0).$$

Again by the coarea formula

$$\int_{\mathbb{R}^d} |f_1(x)|^2 g_1(x) dm^d(x) = \int_{\mathbb{R}} h(s) \int_{\phi_1^{-1}(s)} |f_1(x)|^2 \, |\nabla \phi_1(x)|^{-1} d\mathcal{H}^{d-1}(x) ds.$$

There is a similar formula for the second integral. We assume that f_i is continuous and choose a Dirac sequence for h to obtain the estimate. The statement for measurable functions on the surfaces follows by a standard approximation argument. $\square$

Using the coarea formula we obtain a more explicit formula for the convolution:

$$\begin{aligned}
f_1 h \circ \phi_1 * f_2 h \circ \phi_2(z) &= \int (f_1 h \circ \phi_1)(z-y)(f_2 h \circ \phi_2)(y) dm^d(y) \\
&= \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} h(s) h(t) \int_{\Sigma(s,t)} f_1(z-y) f_2(y) \rho(s,t,z) d\mathcal{H}^{d-2}(y) ds dt.
\end{aligned}$$

There is a trivial improvement

$$\begin{aligned}
\Big\| \int_{\Gamma_1 \cap (z-\Gamma_2)} \gamma^{-1/2}(x,y) \rho(0,0,z) f_1(x) f_2(y) d\mathcal{H}^{d-2} \Big\|_{L^2} \\
\le \|f_1\|_{L^2(\Sigma_1, \delta_{\phi_1})} \|f_2\|_{L^2(\Sigma_2, \delta_{\phi_2})}.
\end{aligned} \tag{4.1}$$

Here $L^2(\Sigma_i, \delta_{\phi_i})$ denotes the space of square integrable functions on the hypersurface with respect to the measure δ_{ϕ_i}.

4.2. **Application to bilinear estimates for dispersive equations.** We use the convolution estimate to bound products of solutions to dispersive equations. Consider

$$iu_t - \psi(D)u = 0.$$

The characteristic surface Σ is a surface in $\mathbb{R}^{d+1}$ defined by the function

$$\phi(\tau,\xi) = \tau - \psi(\xi).$$

The solution defines the measure

$$\hat{u}_0(\xi)\delta_\phi.$$

Let ψ_1 and ψ_2 be real smooth functions and, as above,

$$\phi_1(\tau,\xi) = \tau - \psi_1(\xi) \quad \text{resp. } \phi_2(\tau,\xi) = \tau - \psi_1(\xi).$$

The set of integration is

$$\{(\tilde\tau,\tilde\xi) : \tilde\tau = \tau_2 + \psi_1(\tilde\xi - \xi_2) = \tau_1 + \psi_2(\tilde\xi - \xi_1)\}$$

for $\tau_i = \psi_i(\xi_i)$. We rewrite it with $\xi = \tilde\xi - \xi_1$ and $\tau = \tilde\tau - \tau_1$ as

$$M = \{(\tau,\xi) : \psi_2(\xi) - \psi_1(\xi - (\xi_2 - \xi_1)) = \psi_2(\xi_2) - \psi_1(\xi_2 - (\xi_2 - \xi_1)), \tau = \psi_2(\xi)\}.$$

The most important case will be $\psi_i = \psi$. Then, with $\bar\xi = \xi_2 - \xi_1$,

$$M = \{(\tau,\xi) : \psi(\bar\xi) + \psi(\xi - \bar\xi) - \psi(\xi) = \psi(\bar\xi) + \psi(\xi_2 - \bar\xi) - \psi(\xi_2), \tau = \psi(\xi)\}. \tag{4.2}$$

The expression

$$\psi(\xi_1) + \psi(\xi_2) - \psi(\xi_1 + \xi_2)$$

is called modulation function.

We express the integrand in terms of $\nabla\psi_i$:

$$|\nabla\phi_1|^2|\nabla\phi_2|^2 - (\nabla\phi_1\cdot\nabla\phi_2)^2 = |\nabla\psi_1 - \nabla\psi_2|^2 + |\nabla\psi_1|^2|\nabla\psi_2|^2 - (\nabla\psi_1\cdot\nabla\psi_2)^2. \tag{4.3}$$

The first term is the square of the distance of the gradients, and the second is the square of product of length multiplied by $\sin^2$ of the angle between them.

As a function of ξ this becomes

$$|\nabla\psi_1(\xi - \bar\xi) - \nabla\psi_2(\xi)|^2 + |\nabla\psi_1(\xi-\bar\xi)|^2|\nabla\psi_2(\xi)|^2 - (\nabla\psi_1(\xi-\bar\xi)\cdot\nabla\psi_2(\xi))^2.$$

4.3. **One space dimension.** In one space dimension the second term vanishes. The set $x + \Sigma_1 \cap y + \Sigma_2$ consists generically of a discrete set of points and we obtain a sum of $|\psi_1'(z-x) - \psi_2'(z-y)|$ over the intersection points. Often the intersection consists of one (Schrödinger) or up to two points (Airy). If $\psi(\xi) = \xi^N$ for an even integer N then the equation

$$\xi^N - (\xi - (\xi_2 - \xi_1))^N = \xi_2^N - \xi_1^N$$

has the obvious and unique solution $\xi = \xi_2$. If N is odd there are the exactly two solutions $\xi = \xi_2$ and $\xi = -\xi_1$ unless $\xi_2 = -\xi_1$.

Then

$$\mathcal{F}_t\Big(\int |\psi'(\xi-\eta) - \psi'(\eta)|^{\frac12} e^{it\psi(\xi-\eta)+it\psi(\eta)}\hat{u}_0(\xi-\eta)\hat{u}_1(\eta)d\eta\Big)(\tau)$$
$$= \sum_{(\sigma,\eta)\in\Sigma_1\cap\{(\tau,\xi)-\Sigma_2\}} \gamma^{-1/2}(\xi,\eta)\rho(0,0,z)\hat{u}_0(\eta)\hat{u}_1(\xi-\eta)$$

and its L^2 norm is bounded by 1 (if N is even) and $\sqrt{2}$ (if N is odd) times $\|u_0\|_{L^2}\|u_1\|_{L^2}$.

Theorem 4.2. *With the notation introduced above*

$$\Big\| \int |\psi_0'(\xi-\eta)-\psi_1'(\eta)|^{1/2} e^{it\phi_0(\xi-\eta)+it\phi_1(\eta)} u_0(\xi-\eta)u_1(\eta)d\eta \Big\|_{L^2(\mathbb{R}_t\times\mathbb{R}_\xi)} \le \sqrt{2\pi}\|u_0\|_{L^2(\mathbb{R})}\|u_1\|_{L^2(\mathbb{R})} \tag{4.4}$$

if N is even and if N is odd

$$\Big\| \int |\psi_0'(\xi-\eta)-\psi_1'(\eta)|^{1/2} e^{it\phi_0(\xi-\eta)+it\phi_1(\eta)} u_0(\xi-\eta)u_1(\eta)d\eta \Big\|_{L^2(\mathbb{R}_t\times\mathbb{R}_\xi)} \le \sqrt{2}\sqrt{2\pi}\|u_0\|_{L^2(\mathbb{R})}\|u_1\|_{L^2(\mathbb{R})}. \tag{4.5}$$

There is an interesting special case: Local smoothing corresponds to $\Sigma_1 = \{(\xi^N,\xi)\}$ and Σ_0 is given by $\tau = 0$.

Theorem 4.3. *Let ψ be as above. Then*

$$\||\psi'(D)|^{1/2}S(t)u_0\|_{L^\infty_x L^2_t} \le \sqrt{2\pi}\|u_0\|_{L^2}.$$

if N is odd and if N is even

$$\||\psi'(D)|^{1/2}S(t)u_0\|_{L^\infty_x L^2_t} \le \sqrt{2}\sqrt{2\pi}\|u\|_{L^2}.$$

Proof. We apply the convolution estimate with $\psi_1(\xi) = \xi^N$ and $\psi_0 = 0$. The equation

$$(\xi-\xi_0)^N = -\xi_1^N$$

has the unique solution $\xi = \xi_0 - \xi_1$ if N is odd, and $\xi = \xi_0 \pm \xi_1$ if N is even. Thus, if N is odd

$$\sqrt{N}\int |(|D|^{\frac{N-1}{2}}S(t)u_0)v(x)|^2 dxdt \le \sqrt{2\pi}\|u_0\|^2_{L^2(\mathbb{R})}\|v\|^2_{L^2(\mathbb{R})}$$

and we choose v so that $|v|^2$ is a Dirac sequence. There are only obvious adaptations if N is even. □

In particular, if u satisfies the Airy equation then

$$\|\partial_x u\|_{L^\infty_x L^2(\mathbb{R})} \le \sqrt{2\pi}\|u_0\|_{L^2}$$

and u has square integrable derivatives for almost all t.

4.4. **The Schrödinger equation with higher space dimension.** The characteristic set Σ is a standard parabola.

The set

$$\{(\tau_1,\xi_1)+\Sigma\} \cap (\tau_2,\xi_2)+\Sigma\}$$

is the intersection of two paraboloids. It is paraboloid of dimension $d-1$, the intersection of the paraboloids. The intersection is given by the equations

$$\tau - |\xi_1|^2 = |\xi-\xi_1|^2 \qquad \tau - |\xi_2|^2 = |\xi-\xi_2|^2$$

and hence

$$\tau = |\xi_1|^2 + |\xi-\xi_1|^2 = |\xi_2|^2 + |\xi-\xi_2|^2.$$

The first equality determines τ, which is of minor importance, and the second is equivalent to

$$\langle \xi, \xi_2-\xi_1\rangle = |\xi_2|^2 - |\xi_1|^2 \tag{4.6}$$

which is a hyperplane with normal $\xi_2 - \xi_1$, if $\xi_2 \neq \xi_1$, which we assume for the moment. It contains the point $\xi_2 + \xi_1$, and every point can be written as $\xi_1 + \xi_2 + v$ with v perpendicular to $\xi_2 - \xi_1$. Let

$$w = \frac{\langle \xi_1, \xi_2 - \xi_1 \rangle}{|\xi_2 - \xi_1|^2} \xi_2 - \frac{\langle \xi_2, \xi_2 - \xi_1 \rangle}{|\xi_2 - \xi_1|^2} \xi_1.$$

then

$$|\xi_1|^2 + |v + \xi_2|^2 = |\xi_1|^2 + |\xi_2|^2 - |w|^2 + |v + w|^2 = c_w + |v + w|^2$$

and the area formula

$$\int_M f(\tau, \xi) d\mathcal{H}^{d-1} = \int_{\mathbb{R}^{d-1}} f(c_w + |v + w|^2, \xi_1 + \xi_2 + v) \sqrt{1 + 4|v + w|^2} dv.$$

The integrand is determined by

$$\begin{aligned} J/4 = & \left(|\xi - \xi_2 - (\xi - \xi_1)|^2 + |\xi - \xi_1|^2 |\xi - \xi_2|^2 - \langle \xi - \xi_1, \xi - \xi_2 \rangle^2 \right) \\ = & |\xi_2 - \xi_1|^2 + |v + \xi_2|^2 |v + \xi_1|^2 + \langle v + \xi_2, v + \xi_1 \rangle^2 \\ = & |\xi_2 - \xi_1|^2 (1 + |v + w|^2). \end{aligned}$$

We will choose Σ_1 to be the part of the parabola above the annulus of radii λ and 2λ, and Σ_2 the part of the parabola above the ball of radius μ, with $4\mu \leq \lambda$.

The integrand has size

$$(\lambda^2 + (1 + \lambda^2)|v|^2)^{-1/2}$$

and

$$C(\xi_1, \xi_2) \leq \mu^{d-1} \lambda.$$

Similarly we consider two solutions with the Fourier transform with respect to ξ support in an anulus of size λ, and consider only output of size μ, which corresponds to restricting $|\xi_1 + \xi_2| \leq \mu$.

The integral is then bounded by $\lambda^{-1} \mu^{d-1}$. We obtain the following bilinear estimate

Lemma 4.4 (Schrödinger, d dimensions)**.** *Let $d \geq 2$ and $\mu << \lambda$.*

$$\|u_\lambda v_\mu\|_{L^2} \leq \mu^{\frac{d-1}{2}} \lambda^{-1/2} \|u_\lambda(0)\|_{L^2} \|v_\mu(0)\|_{L^2}$$

$$\|(u_\lambda v_\lambda)_\mu\|_{L^2} \leq \mu^{\frac{d-2}{2}} \|u_\lambda(0)\|_{L^2} \|v_\lambda(0)\|_{L^2}$$

Here $()_\mu$ is defined by the characteristic function of the ball of radius μ as Fourier multiplier, and $()_\lambda$ is the Fourier projection to $\{\xi : |\xi| \geq \lambda\}$.

5. Wellposedness for nonlinear dispersive equations

In this section we will construct local and global solutions to several dispersive equations. The structure of the arguments is always similar.

We study

$$i\partial_t u - \phi(D)u = f(u)$$

where $\phi(D)$ is defined by the real Fourier multiplier $\phi(\xi)$, and f is typically a polynomial in u and its derivatives without constant or linear terms. We denote the linear evolution by $S(t)$ and the adapted U^p and V^p spaces simply by U^p resp. V^p.

5.1. **The (generalized) KdV equation.** For integers $p \geq 1$ we consider the initial value problems

$$u_t + u_{xxx} + (u^p u)_x = 0 \tag{5.1}$$

$$u(0) = u_0 \tag{5.2}$$

- the case $p = 1$ is the Korteweg-de-Vries equation, and $p = 2$ the modified Korteweg-de-Vries equation, and

$$u_t + u_{xxx} + (|u|^p u)_x = 0 \tag{5.3}$$

$$u(0) = u_0 \tag{5.4}$$

for positive real p.

Both equations have soliton solutions

$$u(x,t) = c^{\frac{1}{p}} Q(c^{1/2}(x-ct))$$

with

$$Q_p = \left(\frac{p+1}{2}\right)^{2/p} \cosh^{2/p}(\frac{2}{p}x).$$

The equation is invariant with respect to scaling: $(\lambda^{2/p} u(\lambda x, \lambda^3 t)$ is a solution if u satisfies the equation. The mass $\int u^2 dx$ and energy $\int \frac{1}{2}u_x^2 - \frac{1}{p+2}u^{p+2}$ are conserved. The energy however is not bounded from below.

The space $\dot H^{\frac{1}{2}-\frac{2}{p}}$ (with norm $\|u\|_{\dot H^s} = \||\xi|^s \hat u\|_{L^2}$) is invariant with respect to scaling and it is not hard to see that the generalized KdV equation is globally wellposed in H^1 if $p < 4$. For $p \geq 4$ one expects blow-up. This has been proven in series of seminal papers by Martel, Merle and Martel, Merle and Raphael.

Using the Fourier transform we see that

$$v_t + v_{xxx} = 0 \qquad v(0,x) = v_0(x)$$

defines a unitary group on L^2. We denote

$$S(t)v_0 = v(t)$$

and define the adapted function spaces by

$$\|u\|_{U^p_{KdV}} = \|S(-t)u(t)\|_{U^p}, \qquad \|u\|_{V^p_{KdV}} = \|S(-t)u(t)\|_{V^p},$$

$$\|f\|_{dU^p_{KdV}} = \|S(-t)f(t)\|_{dU^p} \qquad \|f\|_{dV^p_{KdV}} = \|S(-t)u(t)\|_{dV^p}.$$

The Strichartz estimates are

$$\|u\|_{L^p_t L^q_x} \leq c\||D|^{-1/p} u_0\|_{L^2} \tag{5.5}$$

for

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{2}.$$

We have seen they imply the embedding estimates

$$\||D|^{1/p} u\|_{L^p_t L^q_x} \le c\|u\|_{U^p_{KdV}} \tag{5.6}$$

in the same range.

For $\lambda > 0$ we denote

$$u_\lambda = \chi_{\lambda \le |\xi| \le 2\lambda}(D) u$$

the projection of the Fourier transform. Then the Strichartz embedding applied to $g(D)u$, $g(\xi) = |\xi|^{-\frac{1}{p}}$ gives

$$\|u_\lambda\|_{L^p_t L^q_x} \le c\lambda^{-1/p} \|u\|_{U^p_{KdV}} \tag{5.7}$$

- checking atoms one sees that Fourier multipliers act nicely on U^p and V^p.

The bilinear estimates for $4\lambda \le \mu$

$$\|S(t)u_{0,\lambda} S(t) v_{0,\mu}\|_{L^2} \le c\mu^{-1} \|u_{0,\lambda}\|_{L^2} \|v_{0,\mu}\|_{L^2}$$

are a direct consequence of the bilinear estimate of the last section. As for the Strichartz estimate we easily see

$$\|a_\lambda S(t) v_{0,\mu}\|_{L^2} \le c\lambda^{-1} \|v_{0,\mu}\|_{L^2},$$

thus

$$\|u_\lambda S(t) v_{0,\mu}\|_{L^2} \le c\lambda^{-1} \|u_\lambda\|_{U^2_{KdV}} \|v_{0,\mu}\|_{L^2},$$

$$\|u_\lambda a_\mu\|_{L^2} \le c\lambda^{-1} \|u_\lambda\|_{U^2_{KdV}}$$

and

$$\|u_\lambda v_\mu\|_{L^2} \le c\lambda^{-1} \|u_\lambda\|_{U^2_{KdV}} \|v_\mu\|_{U^2_{KdV}}, \tag{5.8}$$

5.1.1. *The case* $p = 3$. We study

$$u_t + u_{xxx} + u_x^4 = 0.$$

Here $\dot H^{-1/6}$ is the critical Sobolev space. We choose a slightly larger space

$$\|u\|_X = \sup_{\lambda \in 2^{\mathbb{Z}}} \lambda^{-1/6} \|u_\lambda\|_{U^2_{KdV}(0,\infty)}$$

for the solution and

$$\|u_0\|_{\dot B^{-1/6}_{2,\infty}} = \sup_{\lambda \in 2^{\mathbb{Z}}} \lambda^{-1/6} \|u_{0,\lambda}\|_{L^2}.$$

Then

$$\sup_\lambda \lambda^{-1/6} \|S(t) u_{0,\lambda}\|_{U^2_{KdV}} \sim \sup_\lambda \lambda^{-1/6} \|u_{0,\lambda}\|_{L^2}$$

There is an ambiguity: By definition $u \in U^p_{KdV}(0,\infty)$ implies $u(0) = 0$. On the other hand we may extend $u(t)$ by zero for negative t. Then $S(t)u_0$ for $t \ge 0$ and 0 for $t < 0$ is a multiple of an atom. We avoid this ambiguity in the formulation of the theorem, but we allow it in its proof.

Theorem 5.1. *There exists $\delta > 0$ such that for all u_0 with*

$$\sup_\lambda \lambda^{-1/6} \|(u_0)_\lambda\|_{L^2} < \delta.$$

there is a unique global solution u which satisfies

$$\|u - S(t)u_0\|_X \le c\|u_0\|_{\dot B^{-1/6}_{2,\infty}}$$

which depends analytically on the initial data.

Proof. We claim

$$\left| \int u_1 u_2 u_3 u_4 v_\lambda dx dt \right| \le \lambda^{-\frac{5}{6}} \prod \|u_i\|_X \|v_\lambda\|_{V^2}. \tag{5.9}$$

Suppose that this estimate is true. We search a solution $u = S(t)u_0 + w$ where

$$w_t + w_{xxx} + (S(t)u_0 + w)^4_x = 0$$

with initial values $w(0) = 0$, which we formulate as fixed point problem of the map $w \to \tilde w$ where

$$\tilde w_t + \tilde w_{xxx} = -(S(t)u_0 + w)^4_x.$$

This equation has to be understood as follows: $\tilde w_\lambda$ satisfies

$$\tilde w_{\lambda,t} + \tilde w_{\lambda,xxx} = (-(S(t)u_0 + w)^4_x)_\lambda$$

in the sense of Lemma 3.33 with $a = 0$ and $b = \infty$. The derivative can be replace by the multiplication by λ after the frequency localization.

By Lemma 3.33 there exists a unique such $w_\lambda \in U^2_{KdV}$ with

$$\lambda^{-1/6} \|w_\lambda\|_{U^2_{KdV}} \le c\|S(t)u_0 + w\|^4_X$$

and, for the difference for two different data

$$\lambda^{-1/6} \|w^2_\lambda - w^1_\lambda\|_{U^2_{KdV}} \le c(\|S(t)u_0 + w^1\|_X + \|S(t)u_0 + w^2\|_X)^3 \|w^2 - w^1\|_X.$$

We take the supremum with respect to λ and arrive at, denoting the map from w to $\tilde w$ by J,

$$\|J(w)\|_X \le c(\|w\|_X + \|u_0\|)^4$$
$$\|J(w^2) - J(w^1)\|_X \le c(\|w^2\|_X + \|w^1\|_X + \|u_0\|)^3 \|w^2 - w^1\|_X.$$

Thus J maps a ball of radius R to a ball of radius

$$c(R + \|u_0\|_{\dot B^{-1/6}_{2,\infty}})^4 < R$$

provided

$$\max\{R^3, \|u_0\|^3_{\dot B^{-1/6}_{2,\infty}}\} < \frac{1}{16c}.$$

Then

$$\|J(w^2) - J(w^1)\|_X \le \frac{1}{2} \|w^2 - w^1\|_X$$

provided $\|w^j\|_X \le R$, $\|u_0\| < \frac{c^{1/3}}{10}$ and $R < \frac{1}{10c^{1/3}}$. We choose $R = \delta = \frac{1}{10c^{1/3}}$. Then J defines a contraction on the closed ball of radius R in X. The contraction mapping theorem implies existence of a unique fixed point, which by Lemma 3.33 is the unique weak solution in X. The map J is a polynom, and hence analytic. The same arguments imply that its derivative is invertible. Now the analytic implicit function theorem in Banach spaces implies an analytic dependence on the initial data.

It remains to prove the estimate

$$\left| \int u_1 u_2 u_3 u_4 v_\lambda dxdt \right| \le \lambda^{-\frac{5}{6}} \prod \|u_i\|_X \|v_\lambda\|_{V^2}. \tag{5.10}$$

We expand the left hand side into a dyadic sum and we try to bound

$$I = \left| \int \prod_{i=1}^{5} u_{i,\lambda_i} dxdt \right|$$

where (by symmetry) $\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4 \le \lambda_5$. We claim

$$\left| \int \prod_{i=1}^{5} u_{i,\lambda_i} dxdt \right| \le c_\varepsilon \lambda_5^{-1} (\lambda_2\lambda_3\lambda_4)^{-1/6} (\lambda_5/\lambda_1)^\varepsilon \prod \|u_{i,\lambda_i}\|_{V^2}. \tag{5.11}$$

We assume that (5.11) holds.

The integral with respect to x vanishes unless there are frequencies in the support of the Fourier transform which add up to zero. Since, if $|\xi_1| \le |\xi_2| \le \dots |\xi_5|$ the frequencies can only add up to zero, $\xi_1+\xi_2+\xi_3+\xi_4+\xi_5 = 0$ if $|\xi_5| - |\xi_1| \ge \frac{1}{10}|\xi_5|$. We replace the decomposition and sum by a finer decomposition $\sum_{\lambda \in 1.01^{\mathbb{Z}}}$ and dyadic decomposition into $\{\xi : \lambda \le \xi < 1.01\lambda\}$. This does change neither the spaces, nor the estimates, up to constants. We observe that $\lambda_4 \ge \lambda_5/8$ - otherwise the integral vanishes. We have to sum over the indices.

(1) $\lambda \ge \lambda_5/10$. Then the factor λ_5^{-1} is eaten up to the derivative and for all λ_4 fixed, the iterated geometric series gives

$$\sum_{\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4} \lambda_1^{1/6-\varepsilon} \le c\lambda_4^\varepsilon$$

which gives the desired bound in this case.

(2) $\lambda = \lambda_3$. Then we sum

$$\sum_{\lambda_1 \le \lambda_2 \le \lambda \le \lambda_4 \sim \lambda_5} \lambda_1^{1/6-\varepsilon} \lambda_3^{5/6} \lambda_5^{-1/3+\varepsilon} \le c\lambda^{1/6}$$

(3) $\lambda = \lambda_2$ and $\lambda = \lambda_1$ is similar.

It remains to prove (5.11) and we have seen that we may assume that $\lambda_1 \le 4\lambda_5/5$. The first attempt then is

$$\begin{aligned} I \le & \|u_{1,\lambda_1} u_{5,\lambda_5}\|_{L^2} \prod_{j=2}^{4} \|u_{j,\lambda_j}\|_{L^6} \\ \le & (\lambda_2\lambda_3\lambda_4)^{-1/6} \lambda_5^{-1} \|u_{1,\lambda_1}\|_{U^2_{KdV}} \|u_{5,\lambda_5}\|_{U^2_{KdV}} \prod_{j=2}^{4} \|u_{j,\lambda_j}\|_{U^6_{KdV}} \end{aligned} \tag{5.12}$$

where we used Hölder's inequality for the first inequality, the bilinear estimate for the first factor, and the L^6 Strichartz embedding for the remaining factors. This is almost what we need - we still have to replace the norm U^2_{KdV} by V^2_{KdV}.

The Strichartz estimates imply

$$\|S(t)u_{0,\lambda} S(t) u_{0,\mu}\|_{L^3} \le c(\lambda\mu)^{-1/6} \|u_{0,\mu}\|_{L^2} \|u_{0,\lambda}\|_{L^2}$$

and the bilinear estimate is - for $\mu \le \lambda/1.03$

$$\|S(t)u_{0,\lambda} S(t) u_{0,\mu}\|_{L^2} \le c\lambda^{-1} \|u_{0,\mu}\|_{L^2} \|u_{0,\lambda}\|_{L^2}$$

provided $\mu \le \lambda/(1.01)^2$. Thus, for $2 \le p \le 3$

$$\|S(t)u_{0,\lambda}S(t)u_{0,\mu}\|_{L^p} \le c\lambda^{-6(\frac{1}{p}-\frac{1}{3})}(\lambda\mu)^{-(\frac{1}{2}-\frac{1}{p})}\|u_{0,\mu}\|_{L^2}\|u_{0,\lambda}\|_{L^2}$$

and hence, by Hölder's inequality

$$\|u_\lambda u_\mu\|_{L^p} \le c\lambda^{2-\frac{5}{p}}\mu^{\frac{1}{p}-\frac{1}{2}}\|u_\mu\|_{U^p_{KdV}}\|u_\lambda\|_{U^p_{KdV}}$$

With this argument we may replace the U^2 by V^2 norms - but now the remaining terms are not square integrable anymore. We use this modified bilinear estimate twice if there are two pairs of λ_i with quotient at least $\ge 1.01^2$. Oversimplifying slightly this leaves us with $\lambda_2 = \lambda_3 \ddot{=} \lambda_5$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ and $\lambda_5 \sim 3\lambda_1$. The second case is easier, and we focus on the first. We again turn our attention to

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 = 0$$

assuming $|\xi_1| \le |\xi_2| \le |\xi_3| \le |\xi_4| \le |\xi_5|$. We have already seen that $|\xi_1| \le 0.9|\xi_5|$. We compose the set $\{\xi : \lambda_j \le |\xi| < 1.01\lambda_j\}$ for $2 \le j \le 5$ into symmetric unions of intervals of length $\lambda_1/100$. We label this intervals by μ_{ij} with $2 \le i \le 5$ and $j \lesssim \lambda_5/\lambda_1$ and expand the sums in

$$\left|\int u_{1,\lambda_1}u_{2,\lambda_5}u_{3,\lambda_5}u_{4,\lambda_5}u_{5,\lambda_5}\,dxdt\right| = \sum_{90 \le |\mu_2+\mu_3+\mu_4+\mu_5| \le 110} \int u_{1,\lambda_1}u_{2,\mu_2}u_{3,\mu_3}u_{4,\mu_4}u_{5,mu_5}\,dxdt$$

there are at most $(\lambda_5/\lambda_1)^4$ terms in the sum- We fix μ_j and assume that they are ordered. Then $\mu_5 - \mu_2 \ge 2$ and we estimate

$$\int u_{1,\lambda_1}u_{2,\mu_2}u_{3,\mu_3}u_{4,\mu_4}u_{5,\mu_5}\,dxdt \le \|u_{\lambda_1}u_{4,\mu_4}\|_{L^p}\|u_{\mu_2}u_{\mu_5}\|_{L^q}\|u_{\lambda_3}\|_{L^6}$$

and hence (changing indices if necessary, or summing over similar terms)

$$\left|\int u_{1,\lambda_1}u_{2,\mu_2}u_{3,\mu_3}u_{4,\mu_4}u_{5,\mu_5}\,dxdt\right| \le c\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-1/6}(\lambda_5/\lambda_1)^5\prod\|u_{i,\lambda_i}\|_{U^p} \tag{5.13}$$

since p is the smallest exponent. This is almost good - but $(\lambda_5/\lambda_1)^5$ is too big.

We recall Lemma 3.11 which allows us to write for given M

$$u = v + w$$

with

$$\frac{\kappa}{M}\|w\|_{U^2_{KdV}} + e^M\|v\|_{U^p_{KdV}} \le \|u\|_{V^2_{KdV}}.$$

We expand all the u_i. This yields by (5.12)

$$\left|\int v_{1,\lambda_1}v_{2,\mu_2}v_{3,\mu_3}v_{4,\mu_4}v_{5,\mu_5}\,dxdt\right| \le cM^5\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-\frac{1}{6}}\prod\|u_{i,\lambda_i}\|_{V^2_{KdV}}$$

and

$$\begin{aligned}\left|\int w_{1,\lambda_1}w_{2,\mu_2}w_{3,\mu_3}w_{4,\mu_4}w_{5,\mu_5}\,dxdt\right| &\le c\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-1/6}(\lambda_5/\lambda_1)^5\prod\|w_{i,\lambda_i}\|_{U^p}\\ &\le e^{-M}\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-1/6}(\lambda_5/\lambda_1)^5\prod\|w_{i,\lambda_i}\|_{V^2_{KdV}}\end{aligned}$$

Similarly we estimate all the other terms in the expansion. Then

$$\begin{aligned}\left|\int u_{1,\lambda_1}u_{2,\mu_2}u_{3,\mu_3}u_{4,\mu_4}u_{5,\mu_5}dxdt\right| \le & c(M^5+e^{-M}(\lambda_5/\lambda_1)^5)\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-1/6} \\ & \times \prod \|u_{i,\lambda_i}\|_{V^2_{KdV}}. \\ \le & \, c\ln(1+(\lambda_5/\lambda_1))^5\lambda_5^{-1}(\lambda_2\lambda_3\lambda_4)^{-1/6}\prod \|u_{i,\lambda_i}\|_{V^2_{KdV}}.\end{aligned}$$

if we choose $M = 5\ln(\lambda_5/\lambda_1)$. This completes the proof of estimate (5.9), and hence the proof of the estimate. □

A variant yields local existence. There are two key oberservations. First we may expand

$$\prod (S(t)u_0+w)_{\lambda_j} = \prod (S(t)u_0)_{\lambda_j} + .. + \prod w_{\lambda_j}$$

there is one term without w, a term linear in w, and higher order terms in w. If w is small than the higher order terms are even smaller. So we need some smallness of the first and the second term. We do not want to assume that the initial data are small, but we are willing to choose a small time.

Theorem 5.2. *Let $R > 0$ there exists $\delta > 0$ so that if*

$$\|u_0\|_{\dot B^{-1/6}_{2,\infty}} \le R$$

and $T > 0$ with

$$\sup_\lambda \|S(t)u_{0,\lambda}\|_{L^6} \le \delta$$

then there is a unique solution u to

$$u_t + u_{xxx} + \partial_x(\chi_{[0,T]}(t)u^4) = 0$$

with initial data u_0 which satisfies

$$\|u - S(t)u_0\|_X \le cR^3\delta^2$$

and which depends analytically on the initial data.

Proof. By the discussion above it suffices to consider integrals

$$\int_0^T \int_{\mathbb{R}} (S(t)u_0)^4 v dxdt.$$

and

$$\int_0^T \int_{\mathbb{R}} (S(t)u_0)^3 wv dxdt.$$

We observe that we may always estimate one $S(t)u_0$ factor in L^6. Thus

$$\|w\|_X \le cR^3\delta$$

which is small provided δ is sufficiently small. □

Here we may have $T = \infty$ even for large initial data. In that case the solution is in U^2_{KdV} and hence

$$w_\lambda = \lim S(-t)u_\lambda(t)$$

exists - since all one sided limits exists. Equivalently

$$u_\lambda(t) - S(t)w_\lambda \to 0$$

in L^2 and the solution to the nonlinear equation is for large t close to a solution to the linear equation. This is called scattering.

Trivially

$$\|\chi_{[0,T]}(t)S(t)u_{0,\lambda}\|_{L^2} \le cT^{1/2}$$

and a simple modification of the scheme above gives local existence for data in L^2, or subcritical Sobolev spaces. Since the L^2 norm is conserved - which needs a proof - there are global solutions for initial data in $L^2(\mathbb{R})$.

Existence of solitons shows that in general solutions are not in L^6 of space-time. Solitons clearly do not scatter.

Wellposedness in a slightly smaller spaces has been proven by Grünrock [9] and Tao [28] based on a modification of the Fourier restriction spaces of Bourgain at the critical level.

Statement and proof are based on [18], where it was one step to prove stability of the soliton in $\dot{B}^{-1/6,2}_{\infty}$, and scattering, which is probably the first stability result of solitons for gKdV which is not based on Weinstein's convexity argument.

5.1.2. *The case $p=4$.* We consider

$$u_t + u_{xxx} + u_x^5 = 0. \tag{5.14}$$

This is the L^2 critical case. We choose

$$\|u_0\|_{B^0_{2,\infty}} = \sup_\lambda \|u_\lambda\|_{L^2}$$

and

$$\|u\|_X = \sup_\lambda \|u_\lambda\|_{U^2_{KdV}}.$$

Theorem 5.3. *There exists $\varepsilon > 0$ such that if*

$$\|u_0\|_{\dot{B}^0_{2,\infty}} < \varepsilon$$

there is a unique global weak solution u in X with

$$\|u - S(t)u_0\|_X \le c\|i_0\|_{\dot{B}^0_{2,\infty}}$$

This time we need an additional inequality called Bernstein's inequality: For $q \ge p$

$$\|u_\lambda\|_q \le \lambda^{\frac{1}{p}-\frac{1}{q}}\|u_\lambda\|_{L^p}. \tag{5.15}$$

Bernstein's inequality is easy to prove. Scaling reduces to question to $\lambda = 1$. So we consider u with Fourier transform supported in $[-2,2]$. We choosea Schwartz function η with $\hat{\eta}(\xi) = 1$ for $|\xi| \le 2$. Then

$$\eta * u_1 = u_1$$

and Youngs inequality gives the bound.

Of Theorem 5.18. Again the assertion follows from the estimate

$$\int u_1u_2u_3u_4u_5v_\lambda dxdt \le c\prod_{i=1}^5 \|u_i\|_X\|v\|_{V^2_{KdV}}$$

in the same fashion as above. We expand the terms and claim

$$\int \prod_{i=1}^6 u_{i,\lambda_i} dxdt \le \lambda_6^{-1+\varepsilon}\lambda_1^{\frac{1}{6}-\varepsilon}(\lambda_3\lambda_4\lambda_5)^{-\frac{1}{6}}.$$

The Strichartz estimate gives

$$\int \prod_{j=1}^{6} u_{j,\lambda_j} dxdt \le \prod \lambda_j^{-1/6} \|u_{j,\lambda_j}\|_{U^6}.$$

The product $\prod_{j=1}^6 \lambda_j^{1/6}$ compensates for the derivative if the output frequency is λ_1, in particular if all frequencies are of the same size.

If $\lambda_6 - \lambda_2 \ge \frac{1}{5}\lambda_6$ - which is the case if λ_1 is much smaller than λ_6, as in the previous case - we estimate

$$\begin{aligned}\int \prod_{j=1}^{6} u_{j,\lambda_j} dxdt \le & \|u_{2,\lambda_2} u_{6,\lambda_6}\|_{L^2} \|u_1\|_{L^\infty} \prod_{j=3}^{5} \|u_{j,\lambda_j}\|_{L^6} \\ \le & \lambda_2^{1/2} (\lambda_3\lambda_4\lambda_5)^{-1/6} \lambda^{-1} \|u_{2,\lambda_2}\|_{U^2_{KdV}} \|u_{6,\lambda_6}\|_{L^2} \|u_{1,\lambda_1}\|_{V^\infty} \prod_{j=3}^{5} \|u_{j,\lambda_j}\|_{U^6}\end{aligned}$$

This is almost good enough, upon replacing U^2 by V^2. For $p > 2$ but close to 2, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{3} = 1$

$$\begin{aligned}\int \prod_{j=1}^{6} u_{j,\lambda_j} dxdt \le & \|u_{2,\lambda_2} u_{6,\lambda_6}\|_{L^p} \|u_1\|_{L^\infty} \|u_3\|_{L^q} \prod_{j=4}^{5} \|u_{j,\lambda_j}\|_{L^6} \\ \le & \lambda_1^{1/2} (\lambda_3\lambda_4\lambda_5)^{-1/6} \lambda_6^{-1} (\lambda_6/\lambda_1)^{\varepsilon} \\ & \times \|u_{2,\lambda_2}\|_{U^p_{KdV}} \|u_{6,\lambda_6}\|_{U^p_{kdV}} \|u_{1,\lambda_1}\|_{V^q_{KdV}} \prod_{j=3}^{5} \|u_{j,\lambda_j}\|_{U^6_{KdV}}.\end{aligned}$$

This is the claimed estimate which completes the proof. □

This version of wellposedness has been proven by Strunk [26]. The result in L^2 is due to Kenig, Ponce and Vega.

5.1.3. *The mKdV equation, $p = 2$.* Here we consider

$$u_t + u_{xxx} + u_x^3 = 0. \tag{5.16}$$

The space $\dot H^{-1/2}$ is scaling invariant, but we are not able the reach the critical space. This time we consider an inhomogeneous space and hence we use a dyadic decomposition with $\lambda \in 2^{\mathbb{N}\cup\{0\}}$ and we define

$$u_\lambda = \chi_{[-1,1]}(D)u$$

for $\lambda = 1$.

This problem is subcritical, and by scaling it suffices to consider small initial data. Wellposedness by different arguments has been shown by [15].

Theorem 5.4. *There exists $\varepsilon > 0$ such that for $u_0 \in B^{\frac14}_{2,\infty}$ with*

$$\|u_0\|_{B^{1/4}_{2,\infty}} \le \varepsilon$$

there is a unique weak solution $u \in X$ with

$$\|u - S(t)u_0\|_X \le c\|u_0\|_{B^{1/4}_{2,\infty}}$$

up to time 1.

Since the problem is subcritical with respect to $B^{1/4}_{2,\infty}$ we can rescale large initial data to small initial data, and we obtain local existence for initial data in $B^{1/4}_{2,\infty}(\mathbb{R})$. We will construct global in time solutions to

$$u_t + u_{xxx} + \partial_x \chi_{[0,T]} u^3 = 0$$

Proof. We want to construct a fixed point of

$$u = \int_0^t S(t-s)\chi_{[0,1]}(s)\partial_x u^3 ds$$

The key estimate (for small data) is

$$|\|\chi_{0,1} u_1 u_2 u_3 \partial_x v_\lambda dxdt| \le \prod \|u_i\|_X \lambda^{-1/4} \|v_\lambda\|_{V^2}. \tag{5.17}$$

We expand into a dyadic sum. The pieces are estimated by

$$\left| \int_0^1 \int_{\mathbb{R}} \prod_{j=1}^4 u_{i,\lambda_i} dxdt \right| \le \prod \lambda_i^{-1/8} \|u_{i,\lambda_i}\|_{U^8_{KdV}} \tag{5.18}$$

by the Strichartz embedding

$$\|u_{i,\lambda_i}\|_{L^8_t L^4_x} \le c\lambda_i^{-1/8} \|u_{i,\lambda_i}\|_{U^8_{KdV}}$$

which we use if $\lambda_1 \sim \lambda_4$ and $\lambda_1 > 1$

$$\left| \int_0^1 \int \prod_{j=1}^4 u_{i,\lambda_i} dxdt \right| \le c\lambda_4^{-2} (\lambda 4/\lambda_1)^\varepsilon \prod \|u_{i,\lambda_i}\|_{V^2_{KdV}} \tag{5.19}$$

if $\lambda_1 \ll \lambda_4$ and $1 < \lambda_4$. The bilinear estimate gives

$$\left| \int_0^1 \int \prod_{j=1}^4 u_{i,\lambda_i} dxdt \right| \le c\lambda_4^{-2} \prod \|u_{i,\lambda_i}\|_{U^2_{KdV}}$$

and we use the previous estimate and Lemma 3.11 to arrive at (5.19). If $\lambda_4 \le 2$ we estimate

$$\begin{aligned} \int_0^1 \int_{\mathbb{R}} \prod_{i=1}^4 u_{i,\lambda_i} dxdt \le & \|u_{1,\lambda_1}\|_{L^\infty} \|u_{2,\lambda_2}\|_{L^\infty} \|u_{3,\lambda_3}\|_{L^\infty L^2} \|u_{4,\lambda_4}\|_{L^\infty L^2} \\ \le & c \prod \|u_{i,\lambda_i}\|_{V^\infty_{KdV}} \end{aligned}$$

We turn to the summation.

(1) $4 \ge \lambda > \lambda_4/2$. This follows using the last estimate.
(2) $\lambda = \lambda_4/2 > 2$ a simple summation gives.

$$\sum_{\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4 = \lambda} \int \prod u_{i,\lambda_i} dxdt \le c(\lambda^{-1} + \lambda^{\varepsilon - 2}) \prod \|u_{i,\lambda_i}\|_{V^2_{KdV}}$$

The same argument works for $\lambda > \lambda_4/100$.

(3) $\lambda_4 = \lambda_2$ or $\lambda_4 = \lambda_1$. This is easier.

The proof is complete. ∎

5.1.4. *The case $p = 2$: the KdV equation.* Here we consider

$$u_t + u_{xxx} + u_x^2 = 0.$$

The well-posedness result in $H^{-\frac{3}{4}}$ is due to Christ, Colliander and Tao [5] who also prove that below $-\frac{3}{4}$ some sort of illposedness must occur. Despite this there are uniform global apriori estimates in H^{-1}, see [3]. Uniqueness between $-\frac{3}{4}$ and -1 is entirely open. Here we have to use an inhomogeneous l^2 Besov summation, and U^2 instead of V^2, i.e

$$\|u\|_X = \left(\sum_{\lambda \in 2^{\mathbb{N}}} \lambda^{-\frac{3}{4}} \|u_\lambda\|_{U^2}^2 + \|u_{<1}(t,x)\|_{X_0} \right)^{1/2}.$$

where the low frequency part has a a complicated structure. It can be written as a sum

$$u{<}\,1 = u^0 + \sum_{\lambda \geq 1} u^\lambda$$

where for $\mu \leq 1$ the Fourier transform of the Fourier projection u_μ^λ is supported in $\mu \leq |\xi| \leq 2\mu$ and

$$|\tau + \xi^3| \geq \mu\lambda^2/100,$$

i.e. it is of high modulation with respect to the frequency λ.

$$\|u_{<1}\|_{X_0} = \inf \left\{ \|u^0\|_{U^2_{KdV}} + \sum_\lambda \left(\sum_{\mu \leq 1} \|u_\mu^\lambda\|_{U^2_{KdV}}^2 \right)^{1/2} + \lambda^{-\frac{1}{2}} \|\partial_x u^\lambda\|_{V^2_{KdV}} \right\}.$$

with the infimum taken over all decompositions of that type.

Theorem 5.5. *There exists $\delta > 0$ such that for all initial data $u_0 \in H^{-\frac{3}{4}}$ with*

$$\|u_0\|_{H^{-\frac{3}{4}}} = \|(1+\xi^2)^{-\frac{3}{8}} \hat{u}_0\|_{L^2}$$

there is a unique solution $u \in X$ with

$$\|u\|_X \leq c\|u_0\|_{H^{-\frac{3}{4}}}$$

up to time 1. *It depends analytically on the initial data.*

Proof. The key estimate is

$$\left| \int_0^1 \int u_1 u_2 \partial_x w \, dx dt \right| \leq c\|u_1\|_X \|u_2\|_X (\sum_{\lambda \in 2^{\mathbb{N}}} \lambda^{3/2} \|w_\lambda\|_{V^2})^{1/2}. \tag{5.20}$$

By duality it allows to set up the fixed point problem. We again expand in the inhomogeneous dyadic frequency ranges. We proceed as above and expand the factors. This times we need a new element, the modulation. We return to the identity

$$\xi_1 + \xi_2 + \xi_3 = 0$$

and

$$\tau_1 + \tau_2 + \tau_3 = 0$$

Since

$$\xi_1^3 + \xi_2^3 - (\xi_1 + \xi_2)^3 = -3\xi_1\xi_2(\xi_1 + \xi_3)$$

the sum of two points on the characteristic set has a distance to the characteristic set described by the modulation function. Given $\lambda_i > 1$ we define the modulation as

$$\Lambda = \lambda_1\lambda_2\lambda_3/100$$

and we decompose

$$u = u^h + u^l$$

where

$$u^l = \chi_{|\tau+\xi^3|<\Lambda}(D_{t,x})u$$

with a smoothed characteristic function χ.

Then the high modulation estimate is

$$\|u^h\|_{L^2} \le c\Lambda^{-1/2}\|u\|_{V^2_{KdV}}.$$

Moreover $S(-t)u^l$ is the convolution of $S(-t)u$ with rescaled Schwartz function, and hence

$$\|u^l\|_{V^p_{KdV}} \le c\|u\|_{V^p_{KdV}}, \quad \|u^h\|_{U^p_{KdV}} \le c\|u\|_{U^p_{KdV}}.$$

The crucial observation is

$$\int u^l_{1,\lambda_1}u^l_{2,\lambda_2}u^l_{3,\lambda_2}\,dxdt = 0$$

since there are no terms in the Fourier support which add up to 0. So at least one term is necessarely of high modulation.

The pair $(1/12, 1/3)$ is a Strichartz pair and hence (for $2 \le \lambda_1 \le \lambda_2 \le \lambda_3$

$$\|u_{\lambda_i}\|_{L^{12}_t L^3_x} \le \lambda_i^{-1/12}\|u\|_{U^{12}}. \tag{5.21}$$

and thus

$$\left|\int_0^1 \int u_{\lambda_1}u_{\lambda_2}u_{\lambda_3}\,dx\,dt\right| \le c(\lambda_1\lambda_2\lambda_3)^{-1/12}\prod \|u_{i,\lambda_i}\|_{U^{12}_{KdV}}. \tag{5.22}$$

As above we see that $\lambda_1 < \frac{4}{5}\lambda_3$ - at least if we refine the dyadic decomposition.

$$\int_0^1 \int_{\mathbb{R}^2} \prod_j u_{\lambda_j}\,dxdt \le \lambda_3^{-1}\|u_{1,\lambda_1}\|_{U^2_{KdV}}\|u_{2,\lambda_2}\|_{V^\infty_{KdV}}\|u_{3,\lambda_3}\|_{U^2_{KdV}} \tag{5.23}$$

by the bilinear estimate. We may also use the decomposition into high and low modulation. Then the most difficult case is

$$\begin{aligned}\left|\int_{\mathbb{R}^2} u^h_{1,\lambda_1}u_{2,\lambda_2}u_{3,\lambda_3}\,dxdt\right| \le& \|u^h_{1,\lambda_1}\|_{L^2}\|(u_{2,\lambda_2}u_{3,\lambda_3})_{\lambda_1}\|_{L^2}\\ \le& c(\lambda_1\lambda_2\lambda_3)^{-1/2}(\lambda_1\lambda_3)^{-1/2}\\ &\times \|u_{1,\lambda_1}\|_{V^2_{KdV}}\|u_{2,\lambda_2}\|_{U^2_{KdV}}\|u_{3,\lambda_3}\|_{U^2_{KdV}}\end{aligned} \tag{5.24}$$

We recall that $\lambda_2 \sim \lambda_3$. Replacing U^2 by V^2 for the integration over $[0,1]$ costs $\ln(1+\lambda_3/\lambda_1)^2$.

The estimate

(5.25)

$$\begin{aligned}\left|\int_{\mathbb{R}^2} u_{1,\lambda_1}u_{2,\lambda_2}u^h_{3,\lambda_3}\,dxdt\right| \le& \|u^h_{3,\lambda_3}\|_{L^2}\|(u_{1,\lambda_1}u_{2,\lambda_2})_{\lambda_3}\|_{L^2}\\ \le& c(\lambda_1\lambda_2\lambda_3)^{-1/2}\lambda_3^{-1}\|u_{3,\lambda_3}\|_{V^2_{KdV}}\|u_{1,\lambda_2}\|_{U^2_{KdV}}\|u_{2,\lambda_2}\|_{U^2_{KdV}}\end{aligned}$$

is stronger. We continue to neglect the case $\lambda_1 = 1$ for a moment. The estimate

$$\left| \int_{\mathbb{R}^2} \chi_{[0,t]} u_{1,\lambda_1} u_{2,\lambda_2} u^h_{3,\lambda_3} dxdt \right| \le c_\varepsilon \lambda_1^{-1} \lambda_3^{-3/2} (\lambda_3/\lambda_1)^\varepsilon \prod \|u_{i,\lambda_i}\|_{V^2} \tag{5.26}$$

suffices for the summation if $\lambda = \lambda_3$, since $\lambda_2 \sim \lambda_3$:

$$\begin{aligned} \sum_{1<\lambda_1<\lambda_3} & \left| \int_{\mathbb{R}^2} \chi_{[0,t]} u_{1,\lambda_1} u_{2,\lambda_3} v^h_{\lambda_3} dxdt \right| \\ & \le c_\varepsilon \lambda_1^{-\frac{1}{4}-\varepsilon} \lambda_3^{\varepsilon-\frac{3}{2}} \|u_1\|_X \|u_{2,\lambda_3}\|_{V^2_{KdV}} \|v_{\lambda_3}\|_{V^2_{KdV}} \end{aligned}$$

and

$$\sum_{\lambda\ge 2} \|u_{2,\lambda}\lambda^{\varepsilon-\frac{1}{2}}\|_{V^2_{KdV}} \|v_{lambda}\|_{V^2_{KdV}} \le C\|u_2\|_X \left(\sum (\lambda^{\frac{3}{4}} \|v_\lambda\|_{V^2_{KdV}})^2 \right)^{1/2}.$$

We consider $\lambda = \lambda_1 \ge 2$. The case of high modulation on the high frequency yields a factor $(\mu/\lambda)^{-1/2+\varepsilon}$ and we restrict to

$$\begin{aligned} \sum_{\mu>\lambda} \left| \int\int v^h_\lambda u_{1,\mu} u_{2,\mu} dxdt \right| \le & \|v_\lambda\|_{V^2_{KdV}} \lambda^{-1} \sum \mu^{-3/2} \|u_{1,\mu}\|_{U^2_{KdV}} \|u_{2,\mu}\|_{U^2_{KdV}} \\ \le & c\|u_1\|_X \|u_2\|_X \lambda^{-1} \|v_\lambda\|_{V^2_{KdV}} \end{aligned}$$

The λ^{-1} compensates the derivative, and

$$\sum_\lambda \|v_\lambda\|_{V^2_{KdV}} \le c \left(\sum_{\lambda\in 1.01^{\mathbb{N}}(\lambda^{\frac{3}{4}}\|v_\lambda} \|_{V^2_{KdV}})^2 \right)^{1/2}.$$

This completes the summation if $\lambda_1 > 1$- since

$$\|\chi_{[0,T]} u\|_{V^p_{KdV}} \le \|u\|_{V^p_{KdV}},$$

and similarly with U^p.

It remains to consider $\lambda_1 = 1$. We expand $v = \sum_{\mu\le 1} v_\mu$. The estimates are stronger with easy summation if the high modulation falls on $u_{2,\lambda}$ or $u_{3,\lambda}$. We set

$$u^0 = \sum_\lambda \sum_\mu (\int_0^t S(t-s)(\partial_x u_{2,\lambda} u_{3,\lambda})_\mu)^l$$

which by the argument above satisfies

$$\|u^0\|_{U^2_{KdV}} \le c\|u_2\|_X \|u_3\|_X$$

If it falls on v^h_μ we use either the high modulation estimate which yields

$$\left| \int_0^1 \int \partial_x v^h_\mu u_{2,\lambda} u_{3,\lambda} dxdt \right| \le c\lambda^{-\frac{3}{2}} \|u_{2,\lambda}\|_{U^2_{KdV}} \|u_{3,\lambda}\|_{U^2_{KdV}} \|v_\mu\|_{V^2_{KdV}} \tag{5.27}$$

if $\lambda^{1/2}\mu \ge 1$. Alternatively

$$\left| \int_0^1 \int \partial_x v^h_\mu u_{2,\lambda} u_{3,\lambda} dxdt \right| \le c\mu\lambda^{-1} \|u_{2,\lambda}\|_{U^2_{KdV}} \|u_{3,\lambda}\|_{U^2_{KdV}} \|v_\mu\|_{U^2} \tag{5.28}$$

which yields for fixed λ a contribution

$$u^\lambda = \left(\int_0^t S(t-s)\chi_{[0,1](s)} \partial_x (u_{2,\lambda} u_{3,\lambda})_{<1} \right)^h .$$

which satisfies

$$\left(\sum_{\mu \le 1} \|u^\lambda_\mu\|^2_{U^2_{KdV}} \right)^{1/2} + \sup_\mu (\mu \lambda^{\frac{1}{2}})^{-1} \|u^\lambda_\mu\|_{V^2_{KdV}} \le c\lambda^{-3/2} \|u_{2,\lambda}\|_{U^2_{KdV}} \|u_{3,\lambda}\|_{U^2_{KdV}} .$$

Altogether, using the embeddings, since the case $\lambda = 1$ is obvious,

$$\|u_{<1}\|_{X^0} \le c\|u_2\|_X \|u_3\|_X . \tag{5.29}$$

We consider

$$\lambda \left| \int_0^1 \int u_{<1} v_\lambda u_{3,\lambda} dx dt \right| .$$

and we decompose $u_{<1}$ according to the definition. A bilinear estimate gives

$$\begin{aligned} \lambda \left| \int_0^1 \int u^0 v_\lambda u_{3,\lambda} dx dt \right| \le & \lambda \|u^0 u_{3,\lambda}\|_{L^2} \|v_\lambda\|_{L^\infty L^2} \\ \le & c \|u^0\|_{U^2_{KdV}} \|u_{3,\lambda}\|_{U^2_{KdV}} \|v_\lambda\|_{V^2_{KdV}} \end{aligned}$$

and the summation with respect to λ poses no difficulty. We fix $\tilde\lambda$ and estimate

$$\lambda \left| \int_0^1 \int u^{\tilde\lambda} v_\lambda u_{3,\lambda} dx dt \right| .$$

Again the easy part is if the high modulation falls on v_λ or $u_{3,\lambda}$, which we leave to the reader. If the high modulation falls on u^λ then its modulation is at least $\mu \max\{\lambda^2, \tilde\lambda^2\}$.

Again the hardest part is the one when the high modulation falls on the first factor. We may ignore the high modulation and use the bilinear estimate for $u^\lambda_\mu u_{2,\lambda}$, which yields

$$\lambda \left| \int_0^1 \int (u^{\tilde\lambda}_\mu)^h u_{2,\lambda} v_\lambda dx dt \right| \le c \|u^{\tilde\lambda}_\mu\|_{U^2_{KdV}} \|u_{2,\lambda}\|_{U^2_{KdV}} \|v_\lambda\|_{V^2_{KdV}}$$

or alternatively

$$\lambda \left| \int_0^1 \int (u^{\tilde\lambda}_\mu)^h u_{2,\lambda} v_\lambda dx dt \right| \le c(\max\{\lambda, \tilde\lambda\}^{1/2}\mu)^{-1} \|u^{\tilde\lambda}_\mu\|_{V^2_{KdV}} \|u_{2,\lambda}\|_{U^2_{KdV}} \|v_\lambda\|_{U^2_{KdV}}$$

and by logarithmic interpolation

$$\lambda \left| \int_0^1 \int (u^{\tilde\lambda}_\mu)^h u_{2,\lambda} v_\lambda dx dt \right| \le c(\max\{\lambda, \tilde\lambda\})^{1/2}\mu)^{\varepsilon - 1} \|u^{\tilde\lambda}_\mu\|_{U^2_{KdV}} \|u_{2,\lambda}\|_{U^2_{KdV}} \|v_\lambda\|_{V^2_{KdV}}$$

We want to use the second estimate if $\mu \ge \min\{\lambda^{-1/2}, \tilde\lambda^{-1/2}$, and the second part of the norm if $\mu \le \tilde\lambda^{-1/2}$. This leads to an easy summation. □

5.2. **The derivative nonlinear Schrödinger equation.** We consider

$$iu_t + \Delta u = \bar{u}\partial_1 \bar{u}. \tag{5.30}$$

This equation has no significance from applications as far as I know. The choice of the nonlinearity is crucial. If u satisfies (5.30) then the same is true for

$$\lambda u(\lambda^2 t, \lambda x)$$

and critical space is $\dot{H}^{\frac{d-2}{2}}$.

The Strichartz with $\frac{2}{4} + \frac{d}{p} = \frac{d}{2}$ and Bernstein, gives for $d \geq 2$

$$\|u_\lambda\|_{L^4(\mathbb{R}\times\mathbb{R}^d)} \leq \lambda^{\frac{d-2}{4}} \|u_{i,\lambda_i}\|_{L^{4,p}(\mathbb{R}^d)} \leq \lambda^{\frac{d-2}{4}} \|u_{i,\lambda_i}\|_{U^4}. \tag{5.31}$$

The bilinear estimates are

$$\|u_\lambda v_\mu\|_{L^2} \leq c\mu^{\frac{d-1}{2}} \lambda^{-1/2} \|u_\lambda\|_{U^2_{i\Delta}} \|v_\mu\|_{U^2_{i\Delta}} \tag{5.32}$$

and

$$\|(u_\lambda v_\lambda)_\mu\|_{L^2} \leq c\mu^{\frac{d-2}{2}} \|u_\lambda\|_{U^2_{i\Delta}} \|v_\lambda\|_{U^2_{i\Delta}}.$$

if $\mu < \lambda/4$. We may improve the second estimate by Bernstein and Strichartz (using a smooth Fourier projection for $_\mu$)

$$\begin{aligned} \|(u_\lambda v_\lambda)_\mu\|_{L^2} \leq & c\mu^{\frac{d-2}{2}} \|u_\lambda v_\lambda\|_{L^2_t L^{\frac{p}{2}}_x} \\ \leq & c\mu^{\frac{d-2}{2}} \|u_\lambda\|_{L^{4,p}} \|v_\mu\|_{L^{4,p}} \\ \leq & c\mu^{\frac{d-2}{2}} \|u_\lambda\|_{U^4_{i\Delta}} \|v_\lambda\|_{U^4_{i\Delta}}. \end{aligned} \tag{5.33}$$

This time we need the complex inner product. The modulation relation is

$$\xi_1^2 + \xi_2^2 + (-\xi_1 - \xi_2)^2 \geq \xi_1^2 + \xi_2^2$$

which is a particularly pleasant situation.

The dyadic estimates become for $\lambda_1 << \lambda_2 \sim \lambda_3$

$$\left| \int u^h_{\lambda_1} u_{\lambda_2} u_{\lambda_3} dx\, dt \right| \leq c\lambda_3^{-1} \lambda_1^{\frac{d-2}{2}} \|u_{1,\lambda_1}\|_{V^2_{i\Delta}} \|u_{2,\lambda_2}\|_{V^2_{i\Delta}} \|u_{3,\lambda_3}\|_{V^2_{i\Delta}} \tag{5.34}$$

and

$$\left| \int u_{\lambda_1} u^h_{\lambda_2} u_{\lambda_3} dx\, dt \right| \leq c\lambda_3^{-\frac{3}{2}} \lambda_1^{\frac{d-1}{2}} \|u_{1,\lambda_1}\|_{U^2_{i\Delta}} \|u_{2,\lambda_2}\|_{V^2_{i\Delta}} \|u_{3,\lambda_3}\|_{U^2_{i\Delta}} \tag{5.35}$$

and hence

$$\left| \int \prod u_{1,\lambda_1} u^h_{2,\lambda_2} u_{3,\lambda_3} dx dt \right| \leq c\lambda_3^{-\frac{3}{2}} \lambda_1^{\frac{d-1}{2}} (\lambda_3/\lambda_1)^\varepsilon \prod_{i=1}^3 \|u_{i,\lambda_i}\|_{V^2_{i\delta}}$$

Theorem 5.6. *Let $d = 2$. There exists $\varepsilon > 0$ so that if*

$$\|u_0\|_{L^2} < \varepsilon$$

then there is a unique solution to

$$iu_t + \Delta u = \bar{u}\partial_{x_1} \bar{u}$$

with

$$\|u\|_X := \left(\sum_{\lambda \in 2^{\mathbb{Z}}} \|u_\lambda\|^2_{U^2_{KdV}} \right)^{1/2} \leq c\|u_0\|_{L^2}.$$

If $d \geq 3$ there exists $\varepsilon > 0$ so that if

$$\|u_0\|_{\dot B^{\frac{d-2}{2}}_{2,1}} = \sum_\lambda \lambda^{\frac{d+2}{2}} \|u_{0,\lambda}\|_{L^2} < \varepsilon$$

then there is a unique weak solution with

$$\|u\|_X := \sum_\lambda \lambda^{\frac{d-2}{2}} \|u_\lambda\|_{U^2} \leq c\|u_0\|_{\dot B^{\frac{d-2}{2}}_{2,1}}$$

Proof. The key estimates are again

$$\left|\int_{\mathbb{R}\times\mathbb{R}^d} (\partial_{x_1}\bar u_1)\bar u_2 \bar v dxdt\right| \leq \|u_1\|_X\|u_2\|_X \left(\sum_\lambda \|v_\lambda\|^2_{V^2_{KdV}}\right)^{1/2}$$

resp.

$$\left|\int_{\mathbb{R}\times\mathbb{R}^d} (\partial_{x_1}\bar u_1)\bar u_2 \bar v dxdt\right| \leq \|u_1\|_X\|u_2\|_X \sup_\lambda \lambda^{-\frac{d-2}{2}} \|v_\lambda\|_{V^2_{KdV}}$$

if $d \geq 3$. We abuse the notation and set $\lambda_2 = \lambda_3 = \lambda$ and compute for $d = 2$

$$\begin{aligned}\sum_{\mu<\lambda} \lambda \left|\int \bar u^h_\mu \bar u_{2,\lambda} \bar v_\lambda dxdt\right| \leq & \sum_{\mu\leq\lambda} \lambda \|u^h_\mu\|_{L^2} \|(u_{2,\lambda}v_\lambda)_\mu\|_{L^2} \\ \leq & \left(\sum_{\mu\leq\lambda} \|u_{1,\mu}\|_{V^2_{i\Delta}})^2\right)^{1/2} \|u_\lambda v_\lambda\|_{L^2(\mathbb{R}^2)} \\ \leq & \|u_1\|_X \|u_\lambda\|_{U^4_{i\Delta}} \|v_\lambda\|_{U^4_{i\Delta}}\end{aligned}$$

The factor λ^{-1} compensates for the derivative. The summation with respect to λ is trivial. The estimate is easier if the high modulation falls on other terms.

$$\begin{aligned}\sum_{\mu<\lambda} \lambda \left|\int \bar u_\mu \bar u^h_{2,\lambda} \bar v_\lambda dxdt\right| \leq & \sum_{\mu\leq\lambda} \lambda \|u^h_{2,\lambda}\|_{L^2} \|u_{1,\mu} v_\lambda\|_{L^2} \\ \leq & \mu^{1/2} \|u_{1,\mu}\|_{V^2_{i\Delta}} \lambda^{-1/2} \|u_{1,\mu}\|_{U^2_{i\Delta}} \|v_\lambda\|_{U^2_{i\Delta}}\end{aligned}$$

By logarithmic interpolation

$$\begin{aligned}\sum_{\mu<\lambda} \lambda \left|\int \bar u_\mu \bar u^h_{2,\lambda} \bar v_\lambda dxdt\right| \leq & \sum_{\mu\leq\lambda} \lambda \|u^h_{2,\lambda}\|_{L^2} \|u_{1,\mu} v_\lambda\|_{L^2} \\ \leq & \sum_{\mu\leq\lambda} (\mu/\lambda)^{\frac12-\varepsilon} \|u_{1,\mu}\|_{V^2_{i\Delta}} \lambda^{-1/2} \|u_{1,\mu}\|_{V^2_{i\Delta}} \|v_\lambda\|_{V^2_{i\Delta}}\end{aligned}$$

and the summation is straight forward.

The modification for $d \geq 3$ is simply: We give up orthogonality and sum for the first estimate

$$\begin{aligned}\sum_{\mu<\lambda} \lambda \left|\int \bar u^h_\mu \bar u_{2,\lambda} \bar v_\lambda dxdt\right| \leq & \sum_{\mu\leq\lambda} \lambda \|u^h_\mu\|_{L^2} \|(u_{2,\lambda}v_\lambda)_\mu\|_{L^2} \\ \leq & \sum_{\mu\leq\lambda} \mu^{\frac{d-2}{2}} \|u_{1,\mu}\|_{V^2_{i\Delta}} \|u_\lambda\|_{V^2_{i\Delta}} \|v_\lambda\|_{V^2_{i\Delta}}\end{aligned}$$

For the second estimate we put in powers of μ resp. λ.

□

References

[1] Georg Aumann. *Reelle Funktionen*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd LXVIII. Springer-Verlag, Berlin, 1954.

[2] Jean Bourgain. Pointwise ergodic theorems for arithmetic sets. *Inst. Hautes Études Sci. Publ. Math.*, (69):5–45, 1989. With an appendix by the author, Harry Furstenberg, Yitzhak Katznelson and Donald S. Ornstein.

[3] Tristan Buckmaster and Herbert Koch. The Korteweg-de-Vries equation at H^{-1} regularity . *arXiv:1112.4657v2*, 2012.

[4] Thierry Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.

[5] Michael Christ, James Colliander, and Terrence Tao. Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. *Amer. J. Math.*, 125(6):1235–1293, 2003.

[6] J. Dieudonné. *Foundations of modern analysis*. Pure and Applied Mathematics, Vol. X. Academic Press, New York, 1960.

[7] Peter K. Friz and Nicolas B. Victoir. *Multidimensional stochastic processes as rough paths*, volume 120 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. Theory and applications.

[8] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.

[9] Axel Grünrock. A bilinear Airy-estimate with application to gKdV-3. *Differential Integral Equations*, 18(12):1333–1339, 2005.

[10] M. Hadac, S. Herr, and H. Koch. Well-posedness and scattering for the KP-II equation in a critical space. *Annales de l'Institut Henri Poincare / Analyse non lineaire*, 2008.

[11] G. H. Hardy and J. E. Littlewood. A convergence criterion for Fourier series. *Math. Z.*, 28(1):612–634, 1928.

[12] B. B. Kadomtsev and V. I. Petviashvili. On the stability of solitary waves in weakly dispersing media. *Sov. Phys., Dokl.*, 15:539–541, 1970.

[13] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[14] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. Oscillatory integrals and regularity of dispersive equations. *Indiana Univ. Math. J.*, 40(1):33–69, 1991.

[15] Carlos E. Kenig, Gustavo Ponce, and Luis Vega. Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Comm. Pure Appl. Math.*, 46(4):527–620, 1993.

[16] H. Koch and D. Tataru. Dispersive estimates for principally normal pseudodifferential operators. *Comm. Pure Appl. Math.*, 58(2):217–284, 2005.

[17] H. Koch and D. Tataru. A Priori Bounds for the 1D Cubic NLS in Negative Sobolev Spaces. *Int. Math. Res. Not.*, 2007:Article ID rnm053, 2007.

[18] Herbert Koch and Jeremy L. Marzuola. Small data scattering and soliton stability in $\dot{H}^{16}$ for the quartic KdV equation. *Analysis and PDE*, 5(1):145–198, 2012.

[19] D. Lépingle. La variation d'ordre p des semi-martingales. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 36(4):295–316, 1976.

[20] Elliott H. Lieb and Michael Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[21] Terry Lyons. Differential equations driven by rough signals. I. An extension of an inequality of L. C. Young. *Math. Res. Lett.*, 1(4):451–464, 1994.

[22] Terry J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana*, 14(2):215–310, 1998.

[23] Jacques Neveu. *Martingales à temps discret*. Masson et Cie, éditeurs, Paris, 1972.

[24] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce National Institute of Standards and Technology, Washington, DC, 2010. With 1 CD-ROM (Windows, Macintosh and UNIX).

[25] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
[26] Nils Strunk. Well-posedness for the supercritical gKdV equation. *arXiv:1209.5206*, 2012.
[27] Terence Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006. Local and global analysis.
[28] Terence Tao. Scattering for the quartic generalised Korteweg-de Vries equation. *J. Differential Equations*, 232(2):623–651, 2007.
[29] S. J. Taylor. Exact asymptotic estimates of Brownian path variation. *Duke Math. J.*, 39:219–241, 1972.
[30] Hans Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.
[31] N. Wiener. The quadratic variation of a function and its Fourier coefficients. *Journ. Mass. Inst. of Technology*, 3:73–94, 1924.
[32] L. C. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.*, 67(1):251–282, 1936.