ARITHMETISCHE GEOMETRIE OBERSEMINAR

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# REAL LOCAL LANGLANDS AS GEOMETRIC LANGLANDS ON THE TWISTOR-$\mathbb{P}^1$

Fargues' program for geometrizing the classical local Langlands correspondence through the Fargues–Fontaine curve has reshaped the conception of the classical local Langlands program. Let $p, \ell$ be two distinct primes, and let $G/\mathbb{Q}_p$ be a reductive group. Instead of asking for a bijection of certain sets of representations as in the classical local Langlands correspondence, a conjecture of Fargues and Scholze [3] postulates an equivalence between

a) the derived category of $\ell$-adic sheaves on the stack $\mathrm{Bun}_G$ of $G$-bundles on the Fargues–Fontaine curve, and
b) (a small modification of) the derived category of quasi-coherent sheaves on the stack of $L$-parameters for the dual group $\hat{G}$.

The aim of this seminar is to study an analog of the Fargues–Scholze conjecture with $\mathbb{Q}_p$ replaced by the archimedean local field $\mathbb{R}$ and with the Fargues–Fontaine curve replaced by the twistor-$\mathbb{P}^1$, i.e., the non-split Brauer-Severi variety $V^+(x^2+y^2+z^2)$ over $\mathbb{R}$. It is reasonable to expect that a picture close to the one discussed in the seminar also exists over $\mathbb{Q}_p$ and with $\mathbb{Q}_p$-coefficients. In comparison to Fargues' classical program taking the ground field $\mathbb{R}$ (resp. $\mathbb{Q}_p$) also as coefficients adds another color to the geometrization picture, namely a profound relation to Hodge theory (resp. to $p$-adic Hodge theory).

In order to faciliate the topic of this ARGOS we briefly discuss important aspects of [3] as far as they are relevant for this seminar:

(1) If $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_p}$ is an affinoid perfectoid space of characteristic $p$, then its relative Fargues–Fontaine curve is an analytic adic space over $\mathbb{Q}_p$ defined by
$$X_S := Y_S/\varphi^{\mathbb{Z}}$$
with $Y_S := \mathrm{Spa}(W(R^+))\backslash V(p[\varpi])$, where $W(-)$ denotes ($p$-typical) Witt vectors, $\varpi \in R$ any pseudo-uniformizer and $\varphi$ the Frobenius. By gluing one can construct the relative Fargues–Fontaine curve for any $S \in \mathrm{Perf}_{\mathbb{F}_p}$.
(2) For a reductive group $G$ over $\mathbb{Q}_p$, the stack sending $S \in \mathrm{Perf}_{\mathbb{F}_p}$ to the groupoid of $G$-bundles on $X_S$ defines the aforementioned small $v$-stack $\mathrm{Bun}_G$.
(3) The functor $S \mapsto H^0(X_S, \mathcal{O})$ is isomorphic to the functor $S \mapsto \underline{\mathbb{Q}_p}(S) := \mathrm{Hom}_{\mathrm{cts}}(|S|, \mathbb{Q}_p)$. From here one deduces the existence of an open immersion $j_1\colon [*/\underline{G(\mathbb{Q}_p)}] \to \mathrm{Bun}_G$ corresponding to the locus of (geometrically fiberwise) trivial $G$-torsors.
(4) A suitably defined derived category of $\ell$-adic sheaves on $\mathrm{Bun}_G$ appears as the "automorphic side" in Fargues' geometrization program. This is motivated by the fact that $\ell$-adic sheaves on $[*/\underline{G(\mathbb{Q}_p)}]$ geometrize smooth representations of the locally profinite group $\underline{G(\mathbb{Q}_p)}$.
(5) The small $v$-sheaf $\mathrm{Div}^1$ is defined as the functor sending $S \in \mathrm{Perf}_{\mathbb{F}_p}$ to the sheaf of Cartier divisors on $X_S$, which are of degree 1. Equivalently, $\mathrm{Div}^1$ is the quotient $\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\}/\mathcal{BC}(\mathcal{O})^\times$ of the punctured Banach–Colmez spaces associated with the line bundles $\mathcal{O}, \mathcal{O}(1)$ on the Fargues–Fontaine curve.

(6) We have $\mathrm{Div}^1 \cong \mathrm{Spa}(\breve{\mathbb{Q}}_p)^\diamond/\varphi^{\mathbb{Z}}$, and in particular $L$-parameters $W_{\mathbb{Q}_p} \to \hat{G}(\mathbb{Q}_\ell)$ can be geometrized as $\hat{G}_{\mathbb{Q}_\ell}$-local systems on $\mathrm{Div}^1$.
(7) The algebraic stack of $\hat{G}$-local systems on $\mathrm{Div}^1$ is the aforementioned stack of $L$-parameters and a suitable version of its derived category of quasi-coherent sheaves defines the "spectral side" in Fargues' program.
(8) Each analytic adic space $Y$ over $\mathbb{Q}_p$ has an associated diamond $f^\diamond \colon Y^\diamond/\varphi^{\mathbb{Z}} \to \mathrm{Div}^1$. The derived pushforward $f^\diamond_*$ of $\ell$-adic étale sheaves calculates étale cohomology while cohomology of $Y^\diamond$ agrees with absolute étale cohomology.

We now indicate the contents of the talks of this seminar, and explain some similarities/differences to Fargues' classical program.

As there is no theory of perfectoid spaces and small $v$-stacks over $\mathbb{R}$, we need the theory of analytic stacks in the sense of [1] for our geometric framework. Some necessary background for this theory will be presented in Talk 3 and Talk 4 after an introductory talk on the classical local Langlands program for $\mathrm{GL}_2(\mathbb{R})$ (Talk 2).

Using the language of analytic stacks the classical Riemann–Hilbert correspondence for a complex manifold $X$ admits a powerful geometric reformulation as an isomorphism $X^{\mathrm{dR}} \cong \underline{X(\mathbb{C})}^{\mathrm{Betti}}$ of the analytic de Rham stack $X^{\mathrm{dR}}$, the quotient of $X$ by the overconvergent neighborhood of the diagonal $\Delta_X \subseteq X \times_{\mathrm{AnSpec}(\mathbb{C})} X$, with the Betti stack of the underlying topological space $X(\mathbb{C})$. Next to the Riemann–Hilbert correspondence Talk 5 discusses as well the analytification of schemes and GAGA - again formulated geometrically as an isomorphism of analytic stacks.

Over $\mathbb{R}$ the critical geometrization of smooth representations of locally profinite groups from (3–4) is replaced by the analysis of the category of quasi-coherent sheaves on the analytic classifying stack $[\mathrm{AnSpec}(\mathbb{C}_{\mathrm{gas}})/G(\mathbb{R})^{\mathrm{la}}]$ of the real-analytic group $G(\mathbb{R})^{\mathrm{la}}$. This is the content of Talk 6. At this place it is worth noting that $\ell$-adic sheaves in the classical case get replaced by plain quasi-coherent sheaves on analytic stacks (and not more complicated theories of coefficients such as $\mathcal{D}$-modules in the geometric Langlands program).

The famous and powerful localization theorem of Beilinson–Bernstein allows to study representations of Lie algebras via the geometry of $\mathcal{D}$-modules on flag varieties. Talk 7 will place this result, and its incarnation for group representations by Kashiwara–Schmid, in the realm of analytic stacks, thereby combining the material of Talks 5 and 6.

In Talk 8 we will start to discuss analogs of Fargues' classical geometrization program by defining relative Fargues–Fontaine curves $X_S$ aka relative twistor spaces, associated with "nil-perfectoid" spaces $S = \mathrm{AnSpec}(A)$. Here, a bounded analytic ring $A$ over $\mathbb{C}_{\mathrm{gas}}$ is called "nil-perfectoid" if its †-reduction $A^{\dagger-\mathrm{red}}$ is the ring of continuous $\mathbb{C}$-valued functions $\mathrm{Cont}(T, \mathbb{C})$ on a compact Hausdorff space $T$.

Talk 9 will discuss a crucial difference between the classical case of Fargues' program and the case over $\mathbb{R}$ studied in this seminar: the functor $S \mapsto H^0(X_S, \mathcal{O})$ on nil-perfectoids is isomorphic to the functor $\mathbb{R}^{\mathrm{la}}$ representing real-analytic functions on $\mathbb{R}$. More generally, this talk will study first examples of "real Banach–Colmez spaces", i.e., functors of sections of vector bundles on relative twistor spaces.

Talk 10 will discuss another crucial difference: quasi-coherent sheaves on the analytic stack $\mathrm{Div}^1$, which is again defined as the moduli of degree 1 Cartier divisors of relative Fargues–Fontaine curves, are not equivalent to representations of the Weil group $W_{\mathbb{R}}$. While vector bundles on $\mathrm{Div}^1_{\mathbb{C}} \cong \mathbb{A}^{2,\mathrm{an}} \setminus \{0\}/W^{\mathrm{la}}_{\mathbb{R}}$ are still equivalent to finite-dimensional continuous complex $W_{\mathbb{R}}$-representations, this is no longer true for families. In particular, there exist interesting new families of $L$-parameters and these families correct the deficiency that the classical local Langlands program over $\mathbb{R}$ does not work in families (as explained in Talk 2).

Point (8) mentioned above admits an interesting analog over $\mathbb{R}$, which yields a new viewpoint on Simpson's variations of twistor structures (Talk 11). Indeed, for a complex manifold $Y$ there exists an analytic stack $Y^{\diamond}$ with a morphism to $\mathrm{Div}^1_{\mathbb{C}}$, and (up to a small modification to be discussed in that talk) vector bundles on $Y^{\diamond}$ are equivalent to variations of twistor structures. This is made possible through the geometry of $Y^{\diamond}$: generically over $\mathrm{Div}^1_{\mathbb{C}}$ the space $Y^{\diamond}$ is the analytic de Rham stack of $Y$, while at the Hodge–Tate point of $\mathrm{Div}^1_{\mathbb{C}}$ the space $Y^{\diamond}$ degenerates to the analytic Hodge–Tate stack of $Y$, which is the classifying stack of the overconvergent tangent bundle of $Y$.

Talk 12 will introduce the "automorphic side" of Fargues' program for $\mathbb{R}$, namely the stack $\mathrm{Bun}_G$ of $G$-bundles on relative Fargues–Fontaine curves/twistor spaces. Following Talk 6 and 9 quasi-coherent sheaves on $\mathrm{Bun}_G$ geometrize locally analytic representations of reductive groups $G$ over $\mathbb{R}$. With $\mathrm{Bun}_G$ at hand, this talk will introduce examples of Hecke operators and discuss an analog of the infinite level Lubin-Tate space and the isomorphism between Lubin-Tate and Drinfeld space at infinite level.

Finally, Talk 13 will take up the Lubin-Tate space of Talk 12 and study its cohomology. This yields a geometric incarnation of the Jacquet-Langlands correspondence and gives a cohomological realization of the $L$-parameters associated to discrete series representations.

**Prerequisites:** Knowledge of [3], Hodge theory or the classical real local Langlands correspondence is not required for following this seminar. Familiarity with [1] or [7] will be very helpful, but major aspects of analytic geometry will be recalled in Talk 3 and Talk 4.

## Talks

### 1. Talk: Overview (9.4.2024)

One of the organizers of this seminar will provide an overview of the seminar, and distribute the remaining talks. Please attend this meeting if you want to give a talk.

### 2. Talk: Classical real local Langlands for $\mathrm{GL}_2(\mathbb{R})$ (16.4.2024)

The aim of this talk is to review the classical local Langlands correspondence for $\mathrm{GL}_2(\mathbb{R})$, i.e., the talk should discuss representations of the Weil group $W_{\mathbb{R}}$, the classification of irreducible, admissible representations up to infinitesimal equivalence and the classical real local Langlands correspondence [8, Part I]. Discuss in detail that the classical correspondence does not work in families [8, Part I]. Give as many details as possible, focus on $\mathrm{GL}_2(\mathbb{R})$ and skip any $\varepsilon$- or $L$-factors. If time permits, discuss irreducible representations of the units of the quaternions $\mathbb{H}^{\times}$ and the Jacquet-Langlands correspondence. Further reference: [4].

### 3. Talk: Analytic rings (23.4.2024)

This talk should introduce analytic rings following [1]. Define light profinite sets, and light condensed sets[1, Lecture 1, Lecture 2]. Discuss the functor from topological spaces to light condensed sets and the equivalence of qcqs light condensed sets with metrizable compact Hausdorff spaces. Define light condensed abelian groups and, if time permits, present a short sketch of the proof that $P := \mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ is internally projective in the category of light condensed abelian groups to highlight the importance of lightness.

Give the definition of an analytic ring $A = (A^{\triangleright}, D(A))$ [1, Lecture 8] (you may state the general animated case), and explain the proposition on good abstract properties of $D(A)$ from that lecture. Mention induced analytic ring structures $(B^{\triangleright}, \mathrm{Mod}_{B^{\triangleright}}(D(A)))$ for an animated $A^{\triangleright}$-algebra $B^{\triangleright} \in D(A)$ as a way of constructing examples. Define the gaseous ring structure on $\mathbb{Z}[q]$, i.e., the analytic ring structure induced by the endomorphism $1 - q \cdot \mathrm{shift}$ on $P \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ [1, Lecture 12]. For the existence you can shortly state that each pre-analytic ring can be turned into an analytic ring [1, Lecture 13]. Discuss the induced examples of solid abelian groups ($q = 1$ in $\mathbb{Z}$, [1, Lecture 5]), only shortly, and the gaseous ring structure on $\mathbb{Z}((q))$ and $\mathbb{R}$ [1, Lecture 14]. Explain how to construct fiber products of analytic rings [1, Lecture 13].

## 4. Talk: Analytic stacks (30.4.2024)

Define the !-topology on $\mathrm{AnRings}^{\mathrm{op}}$ and state the theorem that !-descent implies universal $*$- and !-descent [1, Lecture 16]. In particular, define !-able maps and mention proper morphisms and open immersions of analytic rings [1, Lecture 16]. Explain the examples $\mathrm{AnSpec}(\mathbb{Z}[T], \mathbb{Z}) \to \mathrm{AnSpec}(\mathbb{Z}, \mathbb{Z})$ (a proper morphism) and $\mathrm{AnSpec}(\mathbb{Z}[T], \mathbb{Z}[T]) \to \mathrm{AnSpec}(\mathbb{Z}[T], \mathbb{Z})$ (an open immersion), and discuss the respective lower !-functors (in the non-light setting this is done in [7, Lecture 8]. Define analytic stacks [1, Lecture 19] and discuss the examples in [1, Lecture 19] that occur as well in [8, Part II]. Stop at minute 45, where applications to the Riemann-Hilbert correspondence are given, and mention without details the analytification of schemes (this will be discussed in greater detail in the next talk). Explain the construction of "Betti stacks" for compact Hausdorff spaces (or even light condensed sets) [1, Lecture 15]. Identify their category of quasi-coherent sheaves with sheaves on the underlying topological space [1, Lecture 19, Lecture 20] and identify morphisms $\mathrm{AnSpec}(A) \to K^{\mathrm{Betti}}$ with symmetric monoidal $\otimes$-functors.

## 5. Talk: The Riemann–Hilbert correspondence (7.5.2024)

Construct the norm on $\mathbb{Z}((q))_{\mathrm{gas}}$ [1, Lecture 20] and discuss the analytification of schemes [1, Lecture 20] (a similar discussion can be found in [2, Lecture 6]) and the abstract GAGA-theorem. For analytification, you can restrict to $\mathbb{C}_{\mathrm{gas}}$. Prove the Riemann-Hilbert correspondence as starting at minute 45 in [8, Part II]. Identify the analytic de Rham stack of the affine line as $\mathbb{G}_a^{\mathrm{an}}/\mathbb{G}_a^{\dagger}$. In particular, introduce $\mathbb{G}_a^{\dagger}$. Explain the relation to the classical Riemann–Hilbert correspondence.

## 6. Talk: $D_{\mathrm{qc}}([\mathrm{AnSpec}(\mathbb{C})/G(\mathbb{R})^{\mathrm{la}}])$ (14.5.2024)

The goal of this talk is to discuss the category $D_{\mathrm{qc}}([\mathrm{AnSpec}(\mathbb{C})/G(\mathbb{R})^{\mathrm{la}}])$ and how it gives a good category of $G(\mathbb{R})$-representations. One key point is that $\pi : [\mathrm{AnSpec}(\mathbb{C})/G(\mathbb{R})^{\mathrm{la}}] \to \mathrm{AnSpec}(\mathbb{C})$ is cohomologically smooth with trivializations $f^!(1) \cong 1$ corresponding to choices of Haar measures on $G(\mathbb{R})$. Using this, construct a natural transformation $f^* \to f^!$ where $f : \mathrm{AnSpec}(\mathbb{C}) \to [\mathrm{AnSpec}(\mathbb{C})/G(\mathbb{R})^{\mathrm{la}}]$ is the projection. Then the goal of this talk is to explain how (at least) principal series representations admit canonical definitions in this formalism, under which $f^*$ corresponds to the minimal globalization and $f^!$ to the maximal globalization.

Moreover, prove that for a maximal compact subgroup $K \subset G(\mathbb{R})$, pullback to the classifying stack of $G(\mathbb{R})^{\mathrm{la}} \times_{\underline{G(\mathbb{R})}} \underline{K}$ is fully faithful. Use these ideas to discuss the relation to Harish-Chandra modules.

## No talk on 21.5.2024

## 7. Talk: Beilinson–Bernstein localization (28.5.2024)

The goal of this talk is to prove that, restricting to trivial infinitesimal character in $D_{\mathrm{qc}}([\mathrm{AnSpec}(\mathbb{C})/G(\mathbb{R})^{\mathrm{la}}])$, the category is equivalent to the derived category of sheaves (of gaseous $\mathbb{C}$-vector spaces) on $\mathrm{Fl}(\mathbb{C})/G(\mathbb{R})$. (If time permits, explain the corresponding statement for other infinitesimal characters.) This is a version of the theorem of Kashiwara-Schmid [5]. Discuss the example of $\mathrm{GL}_2$, and in general discuss discrete series representations from this point of view.

## 8. Talk: Relative Fargues–Fontaine curves (4.6.2024)

Define the notion of bounded analytic $\mathbb{C}$-algebras following Rodriguez Camargo [6]. Using this, define the class of "nil-perfectoid" algebras, and define the associated "family of twistor spaces". Prove that every complex- or real-analytic space admits a !-cover by a nil-perfectoid space, and that the whole resulting Čech nerve consists of nil-perfectoids. Discuss that this means that for many purposes, nil-perfectoids can serve as a basis for the topology. Introduce the notion of "real Banach–Colmez spaces", and compute some first examples.

## 9. Talk: Pic and $\mathrm{Div}^1$ (11.6.2024)

Determine the stack of line bundles Pic in this setting; in particular, show that any line bundle has a well-defined (locally constant) degree, and that line bundles of the same degree are locally isomorphic. Using this, define the moduli space of degree 1 Cartier divisors, and compute it explicitly. Use first the presentation in terms of $\mathcal{BC}(\mathcal{O}(1))$, then a second presentation in terms of $\mathcal{BC}(\mathcal{O}(\frac{1}{2}))$. Make the isomorphism between the two presentations explicit.

## 10. Talk: The stack of $L$-parameters (18.6.2024)

The goal of this talk is to dicuss the relation between representations of the real Weil group $W_{\mathbb{R}}$ and vector bundles on $\mathrm{Div}^1$. In particular, show that there are interesting new families of vector bundles on $\mathrm{Div}^1$ that are not seen in the language of $W_{\mathbb{R}}$-representations.

## 11. Talk: Twistor structures and diamonds (25.6.2024)

For any complex manifold $Y$, define the analytic stack $Y^\diamond \to \mathrm{Div}^1$ and discuss its relation to the notion of variations of twistor structures.

## 12. Talk: $\mathrm{Bun}_G$ (2.7.2024)

Define the stack $\mathrm{Bun}_G$ of $G$-bundles in this context, and Hecke operators on them. Discuss in detail the first nontrivial example of a Hecke operator, of minuscule modifications from $\mathcal{O}^2$ to $\mathcal{O}(\frac{1}{2})$. Discuss how this can be used to build a canonical quaternionic torsor over the twistor incarnation of the modular curve.

## 13. Talk: Non-abelian Lubin–Tate theory over $\mathbb{R}$ (9.7.2024)

Using the geometry from the previous talk, discuss the real analogue of the $p$-adic constructions from [9]. In particular, explain how any $G(\mathbb{R})$-representation $\pi$ can be turned into an "infinite-dimensional variation of twistor structures" on $\mathbb{P}^1$, and that it gets cohomology gives a simultaneous realization of the Jacquet–Langlands and local Langlands correspondence, mirroring closely the classical $p$-adic story.

## References


[1] D. Clausen, P. Scholze, *Lectures on analytic stacks*, available at https://www.youtube.com/watch?v=YxSZ1mTIpaA&list=PLx5f8IelFRgGmu6gmL-Kf_Rl_6Mm7juZO

[2] D. Clausen, P. Scholze, *Lectures on complex geometry*, available at https://people.mpim-bonn.mpg.de/scholze/Complex.pdf

[3] L. Fargues, and P. Scholze. *Geometrization of the local Langlands correspondence.* arXiv preprint arXiv:2102.13459 (2021).

[4] A. W. Knapp, *Local Langlands correspondence: the Archimedean case.*, Proc. Sympos. Pure Math. Vol. 55. No. 2. 1994.

[5] M. Kashiwara, and W. Schmid. *Quasi-equivariant D-modules, equivariant derived category, and representations of reductive Lie groups.* Birkhäuser Boston, 1994.

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[7] P. Scholze, *Lectures on condensed mathematics*, available at https://people.mpim-bonn.mpg.de/scholze/Condensed.pdf

[8] P. Scholze, *Emmy-Noether lectures on "Real local Langlands as geometric Langlands on the twistor-$\mathbb{P}^1$"*, available at https://www.youtube.com/watch?v=xqbdkxEhByo, https://www.youtube.com/watch?v=pB7xF4M38XY, https://www.youtube.com/watch?v=GlUkCOJHNSw

[9] P. Scholze, *On the $p$-adic cohomology of the Lubin-Tate tower*, Ann. Sci. Éc. Norm. Supér. (4) 51 (2018), no. 4, 811–863.