The work of Laurent Lafforgue

by Michael Rapoport

Laurent Lafforgue has been awarded the Fields medal for his proof of the Langlands correspondence for the general linear groups GL_r over function fields of positive characteristic. His approach to this problem follows the basic strategy introduced 25 years ago by V. Drinfeld in his proof for GL_2. Already Drinfeld's proof is extremely difficult. Lafforgue's proof is a real tour de force, taking up as it does several hundred pages of highly condensed reasoning. By his achievement Lafforgue has proved himself a mathematician of remarkable strength and perseverance.

In this brief report I will sketch the background of Lafforgue's theorem, state his theorems and then mention some ingredients of his proof. The final passages are devoted to the human factor.

1. The background

The background of Lafforgue's theorem is the web of conjectures known as the *Langlands philosophy* which is a far-reaching generalization of class field theory. Let F be a global field, i.e. either a finite extension of $\mathbf{Q}$ (the *number field case*) or a finite extension of $\mathbf{F}_p(t)$ where $\mathbf{F}_p$ is the finite field with p elements (the *function field case*). Let $\mathbf{A}$ be the adèle ring of F.

Global class field theory may be formulated as giving a bijection between the sets of characters of finite order of the Galois group $\mathrm{Gal}(\bar{F}/F)$ on the one hand, and of the idèle class group $\mathbf{A}^\times/F^\times$ on the other hand. This is the reciprocity law of T. Takagi and E. Artin established in the 1920's as a far-reaching generalization of the quadratic reciprocity law going back to L. Euler. At the end of the 1960's R. Langlands proposed a non-abelian generalization of this reciprocity law. It conjecturally relates the irreducible representations of rank r of $\mathrm{Gal}(\bar{F}/F)$ (or, more generally, of the hypothetical motivic Galois group of F) with cuspidal automorphic representations of $GL_r(\mathbf{A})$. In fact, this conjecture is part of an even grander panorama of Langlands (*the functoriality principle*), in which homomorphisms between L-groups of reductive groups over F induce relations between the automorphic representations on the corresponding groups. These hypothetical reciprocity laws would imply famous conjectures such as the Artin conjecture on the holomorphy of L-functions of irreducible Galois representations, or the Ramanujan-Petersson conjecture on the Hecke eigenvalues of cusp forms for GL_r.

In the number field case, deep results along these lines have been obtained for groups of small rank, such as GL_2, by Langlands himself and many others. And such results have already had spectacular applications such as in the proof of Fermat's last theorem. However, the proof of the Langlands correspondence in any generality in the number field case seems out of reach at the present time. Lafforgue's result, which concerns the function field case, is the first general non-abelian reciprocity law.

2. Lafforgue's theorem

From now on let F denote a function field of characteristic p. We also fix an auxiliary prime number $\ell \neq p$. For a positive integer r let $\mathcal{G}_r$ be the set of equivalence classes of irreducible ℓ-adic representations of dimension r of $\mathrm{Gal}(\bar{F}/F)$. For each $\sigma \in \mathcal{G}_r$, A. Grothendieck defined its L-function $L(\sigma, s)$ which is a rational function in p^{-s} and which satisfies a functional equation of the form $L(\sigma, s) = \varepsilon(\sigma, s) \cdot L(\sigma^\vee, 1-s)$, where $\varepsilon(\sigma, s)$ is a monomial in p^{-s} and where $\sigma^\vee$ denotes the contragredient representation. The L-function is an Euler product, $L(\sigma, s) = \Pi_x L_x(\sigma, s)$, over all places x of F and for a place x of degree $\deg(x)$, where σ is unramified, we have

$$L_x(\sigma, s) = \prod\nolimits_{i=1}^{r} \frac{1}{1 - z_i\, p^{-s\deg(x)}} \quad .$$

Here $z_1, \dots, z_r$ are the Frobenius eigenvalues of σ at x.

Let $\mathcal{A}_r$ be the set of equivalence classes of cuspidal representations of $GL_r(\mathbf{A})$. For each $\pi \in \mathcal{A}_r$, R. Godement and H. Jacquet defined its L-function $L(\pi, s)$ with properties similar to those of the above L-functions. The Euler factor at a place x where π is unramified is given as

$$L_x(\pi, s) = \prod\nolimits_{i=1}^{r} \frac{1}{1 - z_i\, p^{-s\deg(x)}} \quad ,$$

where $z_1, \dots, z_r$ are the Hecke eigenvalues of π at x. The main result of Lafforgue consists of the following theorems.

Theorem 1 (the Langlands conjecture): *There is a bijection $\pi \mapsto \sigma(\pi)$ between $\mathcal{A}_r$ and $\mathcal{G}_r$ characterized by the fact that $L_x(\pi, s) = L_x(\sigma(\pi), s)$ for every place x of F.*

Theorem 2 (the Ramanujan-Petersson conjecture): *Let $\pi \in \mathcal{A}_r$ with central character of finite order. Then for every place x of F where π is unramified, the Hecke eigenvalues $z_1, \dots, z_r \in \mathbf{C}$ are all of absolute value 1.*

Theorem 3 (the Deligne conjecture): *Let $\sigma \in \mathcal{G}_r$ with determinant character of finite order. Then σ is pure of weight 0, i.e. for any place x of F where σ is unramified, the images of the Frobenius eigenvalues $z_1, \dots, z_n$ under any embedding of $\bar{\mathbf{Q}}_\ell$ into $\mathbf{C}$ are of absolute value 1.*

Here Theorems 2 and 3 are consequences of Theorem 1 through P. Deligne's purity theorem and the estimate on Hecke eigenvalues of Jacquet and J. Shalika. Theorem 1 itself is proved by induction on r (*Deligne recursion principle*). After what was known before (in addition to the functional equations, essentially the converse theorems of A. Weil and I. Piatetskii-Shapiro, and the product formula for ε-factors of G. Laumon), it all boiled down to proving the existence of the map $\pi \mapsto \sigma(\pi)$ with the required properties. This is exactly what Lafforgue did.

Before spending a few words on his proof, let us consider the FAQ: What is it good for? The answer is that neither set $\mathcal{G}_r$ or $\mathcal{A}_r$ is simpler than the other in every aspect, but that Theorem 1 can be used to transfer available information in either direction. Theorem

3 is an instance of where information available on $\mathcal{A}_r$ implies results on $\mathcal{G}_r$. In the other direction, Theorem 1 permits one to use constructions available on $\mathcal{G}_r$ to prove various instances of Langlands functoriality for $\mathcal{A}_r$, such as the existence of tensor products, of base change and of automorphic induction.

3. About the proof

The strategy of constructing the map $\pi \mapsto \sigma(\pi)$ is due to Drinfeld and is inspired by the work of Y. Ihara, Langlands and others in the theory of Shimura varieties. It consists in analyzing the ℓ-adic cohomology of the algebraic stack $\mathrm{Sht}_{r,\emptyset}$ over Spec $F \times$ Spec F parametrizing *shtukas of rank* r, resp. the algebraic stack $\mathrm{Sht}_r = \lim_{\leftarrow} \mathrm{Sht}_{r,N}$ parametrizing shtukas of rank r equipped with a compatible system of level structures for all levels N. The latter cohomology module is equipped with an action of $GL_r(\mathbf{A}) \times \mathrm{Gal}(\bar{F}/F) \times \mathrm{Gal}(\bar{F}/F)$ and the aim is to isolate inside it a subquotient of the form

$$\bigoplus\nolimits_{\pi \in \mathcal{A}_r} \pi \otimes \sigma(\pi) \otimes \sigma(\pi)^\vee \quad ,$$

by comparing the Grothendieck-Lefschetz fixed point formula and the Arthur-Selberg trace formula. The essential difficulty is that, in contrast to the case of Shimura varieties, the moduli stack Sht_r is not of finite type — not even at any finite level N. To explain why, recall that a shtuka of rank r is a vector bundle of rank r on X with additional structure (essentially a meromorphic descent datum under Frobenius). Here X is the smooth irreducible projective curve over $\mathbf{F}_p$ with function field F. And, just as the moduli stack of vector bundles of rank r on X is not of finite type, neither are the stacks $\mathrm{Sht}_{r,\emptyset}$ or $\mathrm{Sht}_{r,N}$.

To deal with this difficulty, Lafforgue introduces the open substacks $\mathrm{Sht}_{r,\emptyset}^{\leq P}$ resp. $\mathrm{Sht}_{r,N}^{\leq P}$ of shtukas where the Harder-Narasimhan polygon is bounded by P. These substacks are of finite type and their union is the whole space. The trouble is that they are not stable under the Hecke correspondences. Therefore Lafforgue constructs in the case without level structure a smooth compactification $\overline{\mathrm{Sht}_{r,\emptyset}^{\leq P}}$ of $\mathrm{Sht}_{r,\emptyset}^{\leq P}$ with a normal crossing divisor at infinity, and extends to it the Hecke correspondences by simple normalization. He then applies the Grothendieck-Lefschetz fixed point formula to these correspondences. However, only a part of this formula can be determined explicitly and therefore this seems a pointless exercise. Lafforgue circumvents this problem by isolating the *r-essential part* of the cohomology of $\mathrm{Sht}_{r,\emptyset}^{\leq P}$ and by showing that the remainder, both the difference between the cohomology of $\mathrm{Sht}_{r,\emptyset}^{\leq P}$ and of $\mathrm{Sht}_{r,\emptyset}$ and the cohomology of the boundary of $\overline{\mathrm{Sht}_{r,\emptyset}^{\leq P}}$ is *r-negligible*. Here the work of R. Pink on Deligne's conjecture on the Grothendieck-Lefschetz formula enters in a decisive way. In the case where a level structure is imposed, Lafforgue manages to push through his method by constructing a *partial* compactification of $\mathrm{Sht}_{r,N}^{\leq P}$ which is smooth with a normal crossing divisor at infinity and which is stable under the Hecke correspondences, and by supplementing Pink's theorem by K. Fujiwara's theorem.

4. The months of suspense

Lafforgue's first attempt at a proof of Theorem 1 used a compactification of $\mathrm{Sht}_{r,N}^{\leq P}$. His construction was based on the compactifications of the quotient spaces $X_{r,n} = (PGL_r)^{n+1}/PGL_r$ that he had defined in earlier work, generalizing the case $n = 1$ due to C. De Concini and C. Procesi. In June 2000, while lecturing on his proof, Lafforgue discovered that, contrary to what he had claimed, these compactifications of $X_{r,n}$, and hence also the corresponding compactifications of $\mathrm{Sht}_{r,N}^{\leq P}$, are not smooth in general. He was not even able to resolve their singularities. During two months of suspense in the summer of 2000, Lafforgue managed to fill the gap by finding the above-mentioned partial compactifications of $\mathrm{Sht}_{r,N}^{\leq P}$ and was able to finesse the proof of Theorem 1 from them. Thus in the end, the modified argument is simpler than the original attempt.

Even though Lafforgue's compactifications of $X_{r,n}$ are not used in the final proof, they are fascinating objects in themselves, with close relations to such diverse geometric objects as configuration spaces of matroids, thin Schubert cells, stable degeneration of n-pointed projective lines and local models of Shimura varieties. It turns out that these compactifications are smooth for $n = 1$ resp. $n = 2$ and arbitrary r (De Concini and Procesi, resp. Lafforgue) and for $r = 2$ and arbitrary n (G. Faltings), but can have arbitrarily bad singularities in general (N. Mnev). These compactifications constitute a new field of investigation, taken up by Lafforgue in a 265 page preprint (http://www.ihes.fr/PREPRINTS/M02/Resu/resu-M02-31.html).

5. Biographical data

Laurent Lafforgue was born in 1966. He was a student at the Ecole Normale Supérieure (1986-1990) before entering the Centre National des Recherches Scientifiques in 1990. His academic teacher is Gérard Laumon with whom he obtained his thèse at the Université de Paris-Sud in 1994. It is in the famous *Bâtiment de Mathématique* ("le 425") on the Orsay campus that Lafforgue worked out his proof. Since 2000 he is a professor at the Institut des Hautes Etudes Scientifiques.

6. Further information

For excellent overviews of Lafforgue's proof, cf. Laumon's Bourbaki seminar No. 873, March 2000, which also contains an annotated bibliography, and the notes of Lafforgue of a course at the Tata Institute (http://www.ihes.fr/PREPRINTS/M02/Resu/resu-M02-45.html).

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