Cluster structures on quantum coordinate rings

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Abstract

We show that the quantum coordinate ring of the unipotent subgroup $N(w)$ of a symmetric Kac-Moody group G associated with a Weyl group element w has the structure of a quantum cluster algebra. This quantum cluster structure arises naturally from a subcategory $\mathscr{C}_w$ of the module category of the corresponding preprojective algebra. An important ingredient of the proof is a system of quantum determinantal identities which can be viewed as a q-analogue of a T-system. In case G is a simple algebraic group of type A, D, E, we deduce from these results that the quantum coordinate ring of an open cell of a partial flag variety attached to G also has a cluster structure.

1 Introduction

Let $\mathfrak{g}$ be the Kac-Moody algebra associated with a symmetric Cartan matrix. Motivated by the theory of integrable systems in statistical mechanics and quantum field theory, Drinfeld and Jimbo have introduced its quantum enveloping algebra $U_q(\mathfrak{g})$. Let $\mathfrak{n}$ denote the nilpotent subalgebra arising from a triangular decomposition of $\mathfrak{g}$. In the case when $\mathfrak{g}$ is finite-dimensional, Ringel [**Ri**] showed that the positive part $U_q(\mathfrak{n})$ of $U_q(\mathfrak{g})$ can be realized as the (twisted) Hall algebra of the category of representations over $\mathbb{F}_{q^2}$ of a quiver Q, obtained by orienting the Dynkin diagram of $\mathfrak{g}$. This was a major inspiration for Lusztig's geometric realization of $U_q(\mathfrak{n})$ in terms of Grothendieck groups of categories of perverse sheaves over varieties of representations of Q, which is also valid when $\mathfrak{g}$ is infinite-dimensional [**Lu1**].

The constructions of Ringel and Lusztig involve the choice of an orientation of the Dynkin diagram. In an attempt to get rid of this choice, Lusztig replaced the varieties of representations of Q by the varieties of nilpotent representations of its preprojective algebra $\Lambda = \Lambda(Q)$, which depends only on the underlying unoriented graph. He showed that one can realize the enveloping algebra $U(\mathfrak{n})$ as an algebra of $\mathbb{C}$-valued constructible functions over these nilpotent varieties [**Lu1**, **Lu4**]. The multiplication of $U(\mathfrak{n})$ is obtained as a convolution-type product similar to the product of the Ringel-Hall algebra, but using Euler characteristics of complex varieties instead of number of points of varieties over finite fields. Note that this realization of $U(\mathfrak{n})$ is only available when the Cartan matrix is symmetric.

One of the motivations of this paper was to find a similar construction of the quantized enveloping algebra $U_q(\mathfrak{n})$, as a kind of Ringel-Hall algebra attached to a category of representations of Λ. Unfortunately, there seems to be no simple way of q-deforming Lusztig's realization of $U(\mathfrak{n})$. In this paper we try to overcome this difficulty by switching to the dual picture.

The dual of $U(\mathfrak{n})$ as a Hopf algebra can be identified with the algebra $\mathbb{C}[N]$ of regular functions on the pro-unipotent group N attached to $\mathfrak{n}$. Dualizing Lusztig's construction, one can obtain for each nilpotent representation X of Λ a distinguished regular function $\varphi_X \in \mathbb{C}[N]$, and the product $\varphi_X \varphi_Y$ can be calculated in terms of varieties of short exact sequences with end-terms X and Y [**GLS2**]. For each element w of the Weyl group W of $\mathfrak{g}$, the group N has a finite-dimensional subgroup $N(w)$ of dimension equal to the length of w. In particular, when $\mathfrak{g}$ is finite-dimensional, we have $N = N(w_0)$, where w_0 is the longest element of W. In [**GLS6**], we have shown that the coordinate ring $\mathbb{C}[N(w)]$ is spanned by the functions φ_X where X goes over the objects of a certain subcategory $\mathscr{C}_w$ of mod(Λ). This category was introduced by Buan, Iyama, Reiten and Scott in [**BIRS**], and independently in [**GLS4**] for adaptable w. Moreover, we have proved that $\mathbb{C}[N(w)]$ has a cluster algebra structure in the sense of Fomin and Zelevinsky [**FZ2**], for which the cluster monomials are of the form φ_T for rigid objects T of $\mathscr{C}_w$.

The algebra $\mathbb{C}[N]$ has a quantum deformation, which we denote by $A_q(\mathfrak{n})$ (see below §4.2), and it is well known that the algebras $U_q(\mathfrak{n})$ and $A_q(\mathfrak{n})$ are in fact isomorphic. By works of Lusztig and De Concini-Kac-Procesi, $A_q(\mathfrak{n})$ has a subalgebra $A_q(\mathfrak{n}(w))$, which can be regarded as a quantum deformation of $\mathbb{C}[N(w)]$. On the other hand, Berenstein and Zelevinsky [**BZ2**] have introduced the concept of a quantum cluster algebra. They have conjectured that the quantum coordinate rings of double Bruhat cells in semisimple algebraic groups should have a quantum cluster algebra structure.

In this paper, we introduce for every w an explicit quantum cluster algebra $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$, defined in a natural way in terms of the category $\mathscr{C}_w$. In particular, for every reachable rigid object T of $\mathscr{C}_w$ there is a corresponding quantum cluster monomial $Y_T \in \mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$. Our main result is the following quantization of the above theorem of [**GLS6**].

Theorem 1.1 *There is an algebra isomorphism* $\kappa \colon \mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w) \xrightarrow{\sim} A_q(\mathfrak{n}(w))$.

Note that quantizations of coordinate rings and quantizations of cluster algebras are defined in very different ways. For example $A_q(\mathfrak{n}) \cong U_q(\mathfrak{n})$ is given by its Drinfeld-Jimbo presentation, obtained by q-deforming the Chevalley-Serre-type presentation of $U(\mathfrak{n})$. In contrast, quantum cluster algebras are defined as subalgebras of a skew field of rational functions in q-commuting variables, generated by a usually infinite number of elements given by an inductive procedure.

As a matter of fact, there does not seem to be so many examples of "concrete" quantum cluster algebras in the literature. Grabowski and Launois [**GL**] have shown that the quantum coordinate rings of the Grassmannians $\mathrm{Gr}(2,n)$ ($n \geq 2$), $\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, and $\mathrm{Gr}(3,8)$ have a quantum cluster algebra structure. Lampe [**La1**, **La2**] has proved two particular instances of Theorem 1.1, namely when $\mathfrak{g}$ has type A_n or $A_1^{(1)}$ and $w = c^2$ is the square of a Coxeter element. Recently, the existence of a quantum cluster structure on every algebra $A_q(\mathfrak{n}(w))$ was conjectured by Kimura [**Ki**, Conj.1.1].

Now Theorem 1.1 provides a large class of such examples, including all algebras $U_q(\mathfrak{n})$ for $\mathfrak{g}$ of type A, D, E. By taking $\mathfrak{g} = \mathfrak{sl}_n$ and some special permutation $w_k \in S_n$, one also obtains that the quantum coordinate ring $A_q(\mathrm{Mat}(k, n-k))$ of the space of $k \times (n-k)$-matrices has a quantum cluster algebra structure for every $1 \leq k \leq n$. This may be regarded as a cluster structure on the quantum coordinate ring of an open cell of $\mathrm{Gr}(k,n)$. More generally, for any simply-laced simple algebraic group G, and any parabolic subgroup P of G, we obtain a cluster structure on the quantum coordinate ring of the unipotent radical N_P of P, which can be regarded as a cluster structure on the quantum coordinate ring of an open cell of the partial flag variety G/P.

Note also that, by taking w equal to the square of a Coxeter element, our result gives a Lie theoretic realization of all quantum cluster algebras associated with an arbitrary acyclic quiver (but with a particular choice of coefficients).

Our strategy for proving Theorem 1.1 can be summarized as follows.

Let $A_q(\mathfrak{g})$ be the quantum analogue of the coordinate ring constructed by Kashiwara [**K2**]. We first obtain a general quantum determinantal identity in $A_q(\mathfrak{g})$ (Proposition 3.2). This is a q-analogue (and an extension to the Kac-Moody case) of a determinantal identity of Fomin and Zelevinsky [**FZ1**]. We then transfer this identity to $A_q(\mathfrak{n})$ (Proposition 5.4). Here a little care must be taken since the restriction map from $A_q(\mathfrak{g})$ to $A_q(\mathfrak{n})$ is *not* a ring homomorphism. Appropriate specializations of this identity give rise, for every $w \in W$, to a system Σ_w of equations (Proposition 5.5) allowing to calculate certain quantum minors depending on w in a recursive way. This system is a q-analogue of a T-system arising in various problems of mathematical physics and combinatorics (see [**KNS**]), and we believe it could be of independent interest. It will turn out that all quantum minors involved in Σ_w belong to the subalgebra $A_q(\mathfrak{n}(w))$, and that among them we find a set of algebra generators.

On the other hand, we show that the generalized determinantal identities of [**GLS6**, Theorem 13.1], which relate a distinguished subset of cluster variables, take exactly the same form as the quantum T-system Σ_w when we lift them to the quantum cluster algebra $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$. Therefore, after establishing that the quantum tori consisting of the initial variables of the two systems are isomorphic, we can construct an injective algebra homomorphism κ from $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$ to the skew field of fractions $F_q(\mathfrak{n}(w))$ of $A_q(\mathfrak{n}(w))$, and show that the image of κ contains a set of generators of $A_q(\mathfrak{n}(w))$. Finally, using an argument of specialization $q \to 1$ and our result of [**GLS6**], we conclude that κ is an isomorphism from $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$ to $A_q(\mathfrak{n}(w))$.

Another motivation of this paper was the open orbit conjecture of [**GLS6**, §18.3]. This conjecture states that all functions φ_T associated with a rigid Λ-module T belong to the dual canonical basis of $\mathbb{C}[N]$. It can be seen as a particular instance of the general principle of Fomin and Zelevinsky [**FZ2**] according to which, in every cluster algebra coming from Lie theory, cluster monomials should belong to the dual canonical basis. Since the dual canonical basis of $\mathbb{C}[N]$ is obtained by specializing at $q = 1$ the basis $\mathbf{B}^*$ of $A_q(\mathfrak{n})$ dual to Lusztig's canonical basis of $U_q(\mathfrak{n})$, it is natural to conjecture that, more precisely, every quantum cluster monomial Y_T of $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$ is mapped by κ to an element of $\mathbf{B}^*$. In fact, it is not too difficult to show that $\kappa(Y_T)$ always satisfies one of the two characteristic properties of $\mathbf{B}^*$ (see below §12.3). But unfortunately, the second property remains elusive, although Lampe [**La1**, **La2**] has proved it for all cluster variables in the two special cases mentioned above.

Finally we note that it is well known that the algebras $A_q(\mathfrak{n}(w))$ are skew polynomial rings. Therefore, by Theorem 1.1, all quantum cluster algebras of the form $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$ are also skew polynomial rings, which is far from obvious from their definition. One may hope that, conversely, the existence of a cluster structure on many familiar quantum coordinate rings will bring some new insights for studying their ring-theoretic properties, a very active subject in recent years (see *e.g.* [**BG**, **GLL**, **MC**, **Y**] and references therein).

2 The quantum coordinate ring $A_q(\mathfrak{g})$

2.1 The quantum enveloping algebra $U_q(\mathfrak{g})$

Let $\mathfrak{g}$ be a symmetric Kac-Moody algebra with Cartan subalgebra $\mathfrak{t}$. We follow the notation of [**K2**, §1]. In particular, we denote by I the indexing set of the simple roots α_i $(i \in I)$ of $\mathfrak{g}$, by $P \subset \mathfrak{t}^*$ its weight lattice, by $h_i (i \in I)$ the elements of $P^* \subset \mathfrak{t}$ such that $\langle h_i, \alpha_j \rangle = a_{ij}$ are the entries of the generalized Cartan matrix of $\mathfrak{g}$. Since $\mathfrak{g}$ is assumed to be symmetric, we also have a symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{t}^*$ such that $(\alpha_i, \alpha_j) = a_{ij}$.

The Weyl group $W < \mathrm{GL}(\mathfrak{t}^*)$ is the Coxeter group generated by the reflections s_i for $i \in I$, where

$$s_i(\gamma) := \gamma - \langle h_i, \gamma \rangle \alpha_i.$$

The length of $w \in W$ is denoted by $l(w)$. We will also need the contragradient action of W on P^*.

Let $U_q(\mathfrak{g})$ be the corresponding quantum enveloping algebra, a $\mathbb{Q}(q)$-algebra with generators e_i, f_i $(i \in I), q^h$ $(h \in P^*)$. We write $t_i = q^{h_i}$. We denote by $U_q(\mathfrak{n})$ (*resp.* $U_q(\mathfrak{n}_-)$) the subalgebra of $U_q(\mathfrak{g})$ generated by e_i $(i \in I)$ (*resp.* f_i $(i \in I)$). For $i \in I$, let $U_q(\mathfrak{g}_i)$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, q^h $(h \in P^*)$.

Let M be a (left) $U_q(\mathfrak{g})$-module. For $\lambda \in P$, let $M_\lambda = \{m \in M \mid q^h m = q^{\langle \lambda, h \rangle} m$ for every $h \in P^*\}$ be the corresponding weight space of M. We say that M is *integrable* if (i) $M = \oplus_{\lambda \in P} M_\lambda$, (ii) for any i, M is a direct sum of finite-dimensional $U_q(\mathfrak{g}_i)$-modules, and (iii) for any $m \in M$, there exists $l \geq 0$ such that $e_{i_1} \cdots e_{i_l} m = 0$ for any $i_1, \ldots, i_l \in I$. We denote by $O_{\mathrm{int}}(\mathfrak{g})$ the category of integrable $U_q(\mathfrak{g})$-modules. This is a semisimple category, with simple objects the irreducible highest weight modules $V(\lambda)$ with highest weight $\lambda \in P_+$, the monoid of dominant weights.

2.2 Bimodules

Let φ and $*$ be the $\mathbb{Q}(q)$-linear anti-automorphisms of $U_q(\mathfrak{g})$ defined by

$$\varphi(e_i) = f_i, \quad \varphi(f_i) = e_i, \quad \varphi(q^h) = q^h, \tag{2.1}$$

$$e_i^* = e_i, \quad f_i^* = f_i, \quad (q^h)^* = q^{-h}. \tag{2.2}$$

A right $U_q(\mathfrak{g})$-module N gives rise to a left $U_q(\mathfrak{g})$-module N^φ by defining

$$x \cdot n = n \cdot \varphi(x), \qquad (n \in N, \ x \in U_q(\mathfrak{g})). \tag{2.3}$$

We say that N is an integrable right module if N^φ is an integrable left module. In particular, for $\lambda \in P_+$, we have an irreducible integrable right module $V^{\mathrm{r}}(\lambda)$ such that $V^{\mathrm{r}}(\lambda)^\varphi = V(\lambda)$. Let m_λ be a highest weight vector in $V(\lambda)$, *i.e.* $e_i m_\lambda = 0$ for any $i \in I$. Then m_λ can be regarded as a vector $n_\lambda \in V^{\mathrm{r}}(\lambda)$, which satisfies $n_\lambda f_i = 0$ for any $i \in I$. Equivalently, $V^{\mathrm{r}}(\lambda)$ is isomorphic to the graded dual of $V(\lambda)$, endowed with the natural right action of $U_q(\mathfrak{g})$. It follows that we have a natural pairing $\langle \cdot, \cdot \rangle_\lambda : V^{\mathrm{r}}(\lambda) \times V(\lambda) \to \mathbb{Q}(q)$, which satisfies $\langle n_\lambda, m_\lambda \rangle_\lambda = 1$, and

$$\langle nx, m \rangle_\lambda = \langle n, xm \rangle_\lambda, \qquad (m \in V(\lambda), \ n \in V^{\mathrm{r}}(\lambda), \ x \in U_q(\mathfrak{g})). \tag{2.4}$$

We denote by $O_{\mathrm{int}}(\mathfrak{g}^{\mathrm{op}})$ the category of integrable right $U_q(\mathfrak{g})$-modules. It is also semisimple, with simple objects $V^{\mathrm{r}}(\lambda)$ $(\lambda \in P_+)$.

The tensor product of $\mathbb{Q}(q)$-vector spaces $V^{\mathrm{r}}(\lambda) \otimes V(\lambda)$ has the natural structure of a $U_q(\mathfrak{g})$-bimodule, via

$$x \cdot (n \otimes m) \cdot y = (n \cdot y) \otimes (x \cdot m), \qquad (x, y \in U_q(\mathfrak{g}), \ m \in V(\lambda), \ n \in V^{\mathrm{r}}(\lambda)). \tag{2.5}$$

2.3 Dual algebra

Let $U_q(\mathfrak{g})^* = \mathrm{Hom}_{\mathbb{Q}(q)}(U_q(\mathfrak{g}), \mathbb{Q}(q))$. This is a $U_q(\mathfrak{g})$-bimodule, via

$$(x \cdot \psi \cdot y)(z) = \psi(yzx), \qquad (x, y, z \in U_q(\mathfrak{g}), \ \psi \in U_q(\mathfrak{g})^*). \tag{2.6}$$

On the other hand, $U_q(\mathfrak{g})$ is a Hopf algebra, with comultiplication $\Delta\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ given by

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(q^h) = q^h \otimes q^h, \tag{2.7}$$

with counit $\varepsilon\colon U_q(\mathfrak{g}) \to \mathbb{Q}(q)$ given by

$$\varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \varepsilon(q^h) = 1, \tag{2.8}$$

and with antipode S given by

$$S(e_i) = -t_i^{-1} e_i, \quad S(f_i) = -f_i t_i, \quad S(q^h) = q^{-h}. \tag{2.9}$$

Dualizing Δ, we obtain a multiplication on $U_q(\mathfrak{g})^*$, defined by

$$(\psi \theta)(x) = (\psi \otimes \theta)(\Delta(x)), \qquad (\psi, \theta \in U_q(\mathfrak{g})^*, \ x \in U_q(\mathfrak{g})). \tag{2.10}$$

Later on it will often be convenient to use Sweedler's notation $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ for the co-multiplication. Using this notation, (2.10) reads $(\psi \theta)(x) = \sum \psi(x_{(1)})\theta(x_{(2)})$. Combining (2.6) and (2.10) we obtain

$$x \cdot (\psi \theta) \cdot y = \sum (x_{(1)} \cdot \psi \cdot y_{(1)})(x_{(2)} \cdot \theta \cdot y_{(2)}). \tag{2.11}$$

2.4 Peter-Weyl theorem

Following Kashiwara [**K2**, §7], we define $A_q(\mathfrak{g})$ as the subspace of $U_q(\mathfrak{g})^*$ consisting of the linear forms ψ such that the left submodule $U_q(\mathfrak{g})\psi$ belongs to $O_{\mathrm{int}}(\mathfrak{g})$, and the right submodule $\psi U_q(\mathfrak{g})$ belongs to $O_{\mathrm{int}}(\mathfrak{g}^{\mathrm{op}})$. It follows from the fact that the categories $O_{\mathrm{int}}(\mathfrak{g})$ and $O_{\mathrm{int}}(\mathfrak{g}^{\mathrm{op}})$ are closed under tensor product that $A_q(\mathfrak{g})$ is a subring of $U_q(\mathfrak{g})^*$.

The next proposition of Kashiwara can be regarded as a q-analogue of the Peter-Weyl theorem for the Kac-Moody group G attached to $\mathfrak{g}$ (see [**KP**]). We include a proof for the convenience of the reader.

Proposition 2.1 ([K2, Proposition 7.2.2]) *We have an isomorphism Φ of $U_q(\mathfrak{g})$-bimodules*

$$\bigoplus_{\lambda \in P_+} V^{\mathrm{r}}(\lambda) \otimes V(\lambda) \xrightarrow{\sim} A_q(\mathfrak{g})$$

given by

$$\Phi(n \otimes m)(x) = \langle nx, \, m \rangle_\lambda, \qquad (m \in V(\lambda), \ n \in V^{\mathrm{r}}(\lambda), \ x \in U_q(\mathfrak{g})).$$

Proof — It follows from (2.5) and (2.6) that Φ defines a homomorphism of $U_q(\mathfrak{g})$-bimodules from $\oplus_{\lambda \in P_+} V^{\mathrm{r}}(\lambda) \otimes V(\lambda)$ to $U_q(\mathfrak{g})^*$. Since $V(\lambda)$ and $V^{\mathrm{r}}(\lambda)$ are integrable for all $\lambda \in P_+$, we see that $\mathrm{Im}\,\Phi \subseteq A_q(\mathfrak{g})$.

Let us show that Φ is surjective. Let $\psi \in A_q(\mathfrak{g})$. We want to show that $\psi \in \mathrm{Im}\,\Phi$. Since $V := U_q(\mathfrak{g})\psi$ is integrable, it decomposes as a (finite) direct sum of irreducible integrable modules.

Thus, without loss of generality, we may assume that V is isomorphic to $V(\lambda)$ for some $\lambda \in P_+$. We may also assume that ψ is a weight vector of V (otherwise we can decompose it as a sum of weight vectors). Since $V := U_q(\mathfrak{g})\psi$ and $W := \psi U_q(\mathfrak{g})$ are both integrable, we see that there exist k and l such that $e_{i_1} \cdots e_{i_k} \cdot \psi = \psi \cdot f_{j_1} \cdots f_{j_l} = 0$ for every $i_1, \ldots, i_k, j_1, \ldots, j_l \in I$. Hence $\psi(x e_{i_1} \cdots e_{i_k}) = \psi(f_{j_1} \cdots f_{j_l} x) = 0$ for every $x \in U_q(\mathfrak{g})$. It follows that the linear form $a \in V^*$ defined by $a(\varphi) = \varphi(1)$ takes nonzero values only on a finite number of weight spaces of V. Hence a is in the graded dual of V, which we can identify to $V^r(\lambda)$. Moreover,

$$\Phi(a \otimes \psi)(x) = \langle a, x\psi \rangle_\lambda = a(x\psi) = x\psi(1) = \psi(x)$$

for every $x \in U_q(\mathfrak{g})$. Therefore $\psi = \Phi(a \otimes \psi)$ belongs to $\operatorname{Im}\Phi$.

Now, Φ is also injective. Indeed, if for $n \otimes m \in V^r(\lambda) \otimes V(\lambda)$ we have $\Phi(n \otimes m) = 0$, then for every $x \in U_q(\mathfrak{g})$, $\langle nx, m \rangle_\lambda = 0$. If $n \neq 0$, since $\langle \cdot, \cdot \rangle_\lambda$ is a pairing between $V^r(\lambda)$ and $V(\lambda)$ and $n U_q(\mathfrak{g}) = V^r(\lambda)$, we get that $m = 0$ and $n \otimes m = 0$. Hence the restriction of Φ to $V^r(\lambda) \otimes V(\lambda)$ is injective. Finally, since the bimodules $V^r(\lambda) \otimes V(\lambda)$ are simple and pairwise non-isomorphic, Φ is injective. $\square$

In view of Proposition 2.1, we can think of $A_q(\mathfrak{g})$ as a q-analogue of the coordinate ring of the Kac-Moody group attached to $\mathfrak{g}$ in [**KP**]. We therefore call $A_q(\mathfrak{g})$ the *quantum coordinate ring*. When $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, $A_q(\mathfrak{g})$ is the quantum coordinate ring $\mathcal{O}_q(G)$ of the simply-connected simple Lie group with Lie algebra $\mathfrak{g}$ studied by many authors, see *e.g.* [**J**, §9.1.1].

2.5 Gradings

Let $Q \subset P$ be the root lattice. It follows from the defining relations of $U_q(\mathfrak{g})$ that it is a Q-graded algebra:

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} U_q(\mathfrak{g})_\alpha, \tag{2.12}$$

where

$$U_q(\mathfrak{g})_\alpha = \{x \in U_q(\mathfrak{g}) \mid q^h x q^{-h} = q^{\langle h, \alpha \rangle} x \ \text{for all } h \in P^*\}. \tag{2.13}$$

By Proposition 2.1, we have

$$A_q(\mathfrak{g}) = \bigoplus_{\gamma, \delta \in P} A_q(\mathfrak{g})_{\gamma, \delta}, \tag{2.14}$$

where

$$A_q(\mathfrak{g})_{\gamma, \delta} = \{\psi \in A_q(\mathfrak{g}) \mid q^l \cdot \psi \cdot q^r = q^{\langle r, \gamma \rangle + \langle l, \delta \rangle} \psi \ \text{for all } r, l \in P^*\}. \tag{2.15}$$

Lemma 2.2 *(a) With the above decomposition $A_q(\mathfrak{g})$ is a $P \times P$-graded algebra.*

(b) For $x \in U_q(\mathfrak{g})_\alpha$, $\psi \in A_q(\mathfrak{g})_{\gamma, \delta}$ and $y \in U_q(\mathfrak{g})_\beta$, we have $x \cdot \psi \cdot y \in A_q(\mathfrak{g})_{\gamma - \beta, \delta + \alpha}$.

(c) For $x \in U_q(\mathfrak{g})_\alpha$, $\psi \in A_q(\mathfrak{g})_{\gamma, \delta}$, we have $\psi(x) \neq 0$ only if $\alpha = \gamma - \delta$.

3 Determinantal identities for quantum minors

3.1 Quantum minors

For our convenience we reproduce *mutatis mutandis* a part of [**BZ2**, §9.2]. Using the isomorphism Φ from Proposition 2.1 we define for each $\lambda \in P_+$ the element

$$\Delta^\lambda := \Phi(n_\lambda \otimes m_\lambda) \in A_q(\mathfrak{g})_{\lambda,\lambda}. \tag{3.1}$$

This is a q-analogue of a (generalized) principal minor, in the sense of [**FZ1**, §1.4]. An easy calculation shows that

$$\Delta^\lambda(f q^h e) = \varepsilon(f) q^{\langle h,\lambda \rangle} \varepsilon(e), \qquad (f \in U_q(\mathfrak{n}_-),\ h \in P^*,\ e \in U_q(\mathfrak{n})). \tag{3.2}$$

For $(u,v) \in W \times W$, we choose reduced expressions $\mathbf{i} = (i_{l(u)},\ldots,i_2,i_1)$ and $\mathbf{j} = (j_{l(v)},\ldots,j_2,j_1)$ so that $u = s_{i_{l(u)}} \cdots s_{i_2} s_{i_1}$ and $v = s_{j_{l(v)}} \cdots s_{j_2} s_{j_1}$. Next, we introduce positive roots

$$\beta_k = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}), \quad \gamma_l = s_{j_1} s_{j_2} \cdots s_{j_{l-1}}(\alpha_{j_l}), \qquad (1 \le k \le l(u),\ 1 \le l \le l(v)). \tag{3.3}$$

Finally, for $\lambda \in P_+$, we set

$$b_k = (\beta_k, \lambda), \quad c_l = (\gamma_l, \lambda), \qquad (1 \le k \le l(u),\ 1 \le l \le l(v)), \tag{3.4}$$

and we define the (generalized) *quantum minor* $\Delta_{u(\lambda),v(\lambda)} \in A_q(\mathfrak{g})$ by

$$\Delta_{u(\lambda),v(\lambda)} = \left(f_{j_{l(v)}}^{(c_{l(v)})} \cdots f_{j_1}^{(c_1)} \right) \cdot \Delta^\lambda \cdot \left(e_{i_1}^{(b_1)} \cdots e_{i_{l(u)}}^{(b_{l(u)})} \right). \tag{3.5}$$

Here, as usual, we denote by $e_i^{(k)}$ (*resp.* $f_i^{(k)}$) the q-divided powers of the Chevalley generators. Equivalently, for $x \in U_q(\mathfrak{g})$ we have

$$\Delta_{u(\lambda),v(\lambda)}(x) = \Delta^\lambda \left(e_{i_1}^{(b_1)} \cdots e_{i_{l(u)}}^{(b_{l(u)})} x f_{j_{l(v)}}^{(c_{l(v)})} \cdots f_{j_1}^{(c_1)} \right). \tag{3.6}$$

It follows from the quantum Verma relations [**Lu2**, Proposition 39.3.7] that $\Delta_{u(\lambda),v(\lambda)}$ depends only on the pair of weights $(u(\lambda), v(\lambda))$, and not on the choice of u and v, or of their reduced expressions $\mathbf{i}$ and $\mathbf{j}$. Moreover it is immediate that $\Delta_{u(\lambda),v(\lambda)} \in A_q(\mathfrak{g})_{u(\lambda),v(\lambda)}$.

We have the following direct consequence of the definition of quantum minors.

Lemma 3.1 *If $l(s_i u) = l(u) + 1$ and $l(s_j v) = l(v) + 1$ then*

$$\Delta_{s_i u(\lambda), s_j v(\lambda)} = f_j^{(c)} \cdot \Delta_{u(\lambda),v(\lambda)} \cdot e_i^{(b)},$$

where $b := (\alpha_i, u(\lambda)) \ge 0$ (resp. $c := (\alpha_j, v(\lambda)) \ge 0$) is the maximal natural number such that $\Delta_{u(\lambda),v(\lambda)} \cdot e_i^{(b)} \ne 0$ (resp. $f_j^{(c)} \cdot \Delta_{u(\lambda),v(\lambda)} \ne 0$).

It is convenient to identify $\Delta_{u(\lambda),v(\lambda)}$ with an extremal vector of weight $(u(\lambda), v(\lambda))$ in the simple $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ highest weight module $V^r(\lambda) \otimes V(\lambda)$. Thus, we have

$$f_i \cdot \Delta_{\gamma,\delta} = 0 \text{ if } (\alpha_i, \delta) \le 0, \qquad e_i \cdot \Delta_{\gamma,\delta} = 0 \text{ if } (\alpha_i, \delta) \ge 0, \tag{3.7}$$

$$\Delta_{\gamma,\delta} \cdot e_i = 0 \text{ if } (\alpha_i, \gamma) \le 0, \qquad \Delta_{\gamma,\delta} \cdot f_i = 0 \text{ if } (\alpha_i, \gamma) \ge 0. \tag{3.8}$$

In particular, we have $f_{j_{l(v)}} \cdot \Delta_{u(\lambda),v(\lambda)} = 0 = \Delta_{u(\lambda),v(\lambda)} \cdot e_{j_{l(u)}}$.

One may also identify $\Delta_{u(\lambda),v(\lambda)}$ with a matrix coefficient in $V(\lambda)$. To do that, let us first denote by

$$m_{v(\lambda)} := f_{j_{l(v)}}^{(c_{l(v)})} \cdots f_{j_1}^{(c_1)} m_\lambda \qquad (3.9)$$

the extremal weight vector of $V(\lambda)$ with weight $v(\lambda)$. Next, let $(\cdot, \cdot)_\lambda$ be the nondegenerate bilinear form on $V(\lambda)$ defined by

$$(xm_\lambda, ym_\lambda)_\lambda := \langle n_\lambda \varphi(x), ym_\lambda \rangle_\lambda, \qquad (x,y \in U_q(\mathfrak{g})). \qquad (3.10)$$

Then, using (3.1) and (3.6) we easily get

$$\Delta_{u(\lambda),v(\lambda)}(x) = (m_{u(\lambda)}, xm_{v(\lambda)})_\lambda, \qquad (x \in U_q(\mathfrak{g})). \qquad (3.11)$$

3.2 A family of identities for quantum minors

Let $(\varpi_i)_{i \in I}$ be the fundamental weights, $i.e.$ we have $\langle h_j, \varpi_i \rangle = (\alpha_j, \varpi_i) = \delta_{ij}$. We note that the fundamental weights are only determined up to a W-invariant element. Moreover, it is useful to observe that

$$s_i(\varpi_j) = \begin{cases} \varpi_j - \alpha_j & \text{if } i = j, \\ \varpi_j & \text{otherwise.} \end{cases} \qquad (3.12)$$

We can now state the main result of this section. This is a q-analogue of [**FZ1**, Theorem 1.17].

Proposition 3.2 *Suppose that for $u, v \in W$ and $i \in I$ we have $l(us_i) = l(u) + 1$ and $l(vs_i) = l(v) + 1$. Then*

$$\Delta_{us_i(\varpi_i),vs_i(\varpi_i)} \Delta_{u(\varpi_i),v(\varpi_i)} = q^{-1} \Delta_{us_i(\varpi_i),v(\varpi_i)} \Delta_{u(\varpi_i),vs_i(\varpi_i)} + \prod_{j \neq i} \Delta_{u(\varpi_j),v(\varpi_j)}^{-a_{ji}} \qquad (3.13)$$

holds in $A_q(\mathfrak{g})$.

The proof of this proposition will be given after some preparation in Section 3.3 below, by following essentially the strategy of [**FZ1**, §2.3]. We first continue to review material from [**BZ2**, §9.2].

Lemma 3.3 *Let $\psi \in A_q(\mathfrak{g})_{\gamma,\delta}$ and $\psi' \in A_q(\mathfrak{g})_{\gamma',\delta'}$. For $i \in I$, assume that $a = (\alpha_i, \delta)$ and $a' = (\alpha_i, \delta')$ are the maximal non-negative integers such that $f_i^a \cdot \psi \neq 0$ and $f_i^{a'} \cdot \psi' \neq 0$. Then*

$$(f_i^{(a)} \cdot \psi)(f_i^{(a')} \cdot \psi') = f_i^{(a+a')} \cdot (\psi \psi').$$

This follows from the definition of the comultiplication of $U_q(\mathfrak{g})$ and from (2.11). Note that we have an analogous result where f_i is replaced by e_i and acts from the right. The next lemma is an immediate consequence of Lemma 3.1 and Lemma 3.3.

Lemma 3.4 *Let $\lambda', \lambda'' \in P_+$, $u', v', u'', v'' \in W$, and $i, j \in I$, be such that*

$$l(s_j v') = l(v') + 1, \quad l(s_j v'') = l(v'') + 1, \quad l(s_i u') = l(u') + 1, \quad l(s_i u'') = l(u'') + 1.$$

Then, putting $a = (\alpha_j, v'(\lambda') + v''(\lambda''))$, and $b = (\alpha_i, u'(\lambda') + u''(\lambda''))$, we have

$$f_j^{(a)} \cdot (\Delta_{u'(\lambda'),v'(\lambda')} \Delta_{u''(\lambda''),v''(\lambda'')}) = \Delta_{u'(\lambda'),s_j v'(\lambda')} \Delta_{u''(\lambda''),s_j v''(\lambda'')},$$

$$(\Delta_{u'(\lambda'),v'(\lambda')} \Delta_{u''(\lambda''),v''(\lambda'')}) \cdot e_i^{(b)} = \Delta_{s_i u'(\lambda'),v'(\lambda')} \Delta_{s_i u''(\lambda''),v''(\lambda'')}.$$

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The next lemma follows easily from Lemma 3.4 by induction on the length of u and v.

Lemma 3.5 *The quantum minors have the following multiplicative property:*

$$\Delta_{u(\lambda),v(\lambda)}\,\Delta_{u(\mu),v(\mu)} = \Delta_{u(\lambda+\mu),v(\lambda+\mu)}, \quad (u,v \in W,\ \lambda,\mu \in P_+).$$

In particular, the factors of the second summand in the right hand side of (3.13) pairwise commute.

The factors of the first summand in the right hand side of (3.13) also commute with each other, as shown by the following lemma.

Lemma 3.6 *Suppose that for $u,v \in W$ and $i \in I$ we have $l(us_i) = l(u)+1$ and $l(vs_i) = l(v)+1$. Then*

$$\Delta_{us_i(\varpi_i),v(\varpi_i)}\,\Delta_{u(\varpi_i),vs_i(\varpi_i)} = \Delta_{u(\varpi_i),vs_i(\varpi_i)}\,\Delta_{us_i(\varpi_i),v(\varpi_i)}.$$

Proof — For $x \in U_q(\mathfrak{n})$ we have

$$x \cdot \Delta_{s_i(\varpi_i),\varpi_i} = x \cdot \Delta^{\varpi_i} \cdot e_i = \varepsilon(x)\Delta^{\varpi_i} \cdot e_i = \varepsilon(x)\Delta_{s_i(\varpi_i),\varpi_i}.$$

Similarly, for $y \in U_q(\mathfrak{n}_-)$ we have

$$\Delta_{\varpi_i,s_i(\varpi_i)} \cdot y = \varepsilon(y)\Delta_{\varpi_i,s_i(\varpi_i)}.$$

By [**BZ2**, Lemma 10.2] we can then conclude that

$$\Delta_{s_i(\varpi_i),\varpi_i}\,\Delta_{\varpi_i,s_i(\varpi_i)} = \Delta_{\varpi_i,s_i(\varpi_i)}\,\Delta_{s_i(\varpi_i),\varpi_i}.$$

(In [**BZ2**], $\mathfrak{g}$ is generally assumed to be finite-dimensional, but this assumption plays no role in the proof of [**BZ2**, Lemma 10.2].) This proves the lemma for $u = v = e$. The general case then follows from successive applications of Lemma 3.4. $\qquad\square$

3.3 Proof of Proposition 3.2

With the help of Lemma 3.4, we see by an easy induction on the length of u and v that it is sufficient to verify the special case $u = v = e$, i.e.

$$\underbrace{\Delta_{s_i(\varpi_i),s_i(\varpi_i)}\,\Delta_{\varpi_i,\varpi_i} - q^{-1}\Delta_{s_i(\varpi_i),\varpi_i}\,\Delta_{\varpi_i,s_i(\varpi_i)}}_{\psi_1} = \underbrace{\prod_{j\neq i}\Delta_{\varpi_j,\varpi_j}^{-a_{ji}}}_{\psi_2}. \qquad (3.14)$$

Note that

$$\gamma_i := -\sum_{j\neq i}a_{ji}\varpi_j = 2\varpi_i - \alpha_i = \varpi_i + s_i(\varpi_i) \in P_+.$$

Thus, by Lemma 3.5 we have

$$\psi_2 = \Delta^{\gamma_i} = \Delta_{\gamma_i,\gamma_i}, \qquad (3.15)$$

in particular the factors of ψ_2 commute. It follows from the definition of Δ^{γ_i} that

(1) $\psi_2(1_U) = 1$,

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(2) $\psi_2 \in A_q(\mathfrak{g})_{\gamma_i, \gamma_i}$,

(3) $e_j \cdot \psi_2 = 0 = \psi_2 \cdot f_j$ for all $j \in I$.

Here 1_U stands for the unit of $U_q(\mathfrak{g})$. By Proposition 2.1 these properties characterize ψ_2 uniquely, so it is sufficient to verify the properties (1) – (3) for ψ_1.

Property (1). We use that for any $\psi, \phi \in A_q(\mathfrak{g})$ we have $(\psi\phi)(1_U) = \psi(1_U)\phi(1_U)$ since $\Delta(1_U) = 1_U \otimes 1_U$, and note that $q^0 = 1_U$. Now, $\Delta_{\varpi_i, s_i(\varpi_i)} = f_i \cdot \Delta^{\varpi_i}$, thus $\Delta_{\varpi_i, s_i(\varpi_i)}(1_U) = \Delta^{\varpi_i}(f_i) = 0$. Similarly, $\Delta_{s_i(\varpi_i), \varpi_i}(1_U) = 0$ and $\Delta_{\varpi_i, \varpi_i}(1_U) = 1$. Finally,

$$\Delta_{s_i(\varpi_i), s_i(\varpi_i)}(1_U) = \Delta^{\varpi_i}(e_i f_i) = \Delta^{\varpi_i}(f_i 1_U e_i) + \Delta^{\varpi_i}\left(\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}\right) = 0 + 1.$$

Thus $\psi_1(1_U) = 1$.

Property (2). Recall that $\Delta_{\gamma, \delta} \in A_q(\mathfrak{g})_{\gamma, \delta}$. Since $A_q(\mathfrak{g})$ is $P \times P$-graded it follows from (3.15) that both summands of ψ_1 belong to $A_q(\mathfrak{g})_{\gamma_i, \gamma_i}$.

Property (3). Since $\Delta(e_j) = e_j \otimes 1 + q^{h_j} \otimes e_j$ the operator $e_j \cdot -$ acts on $A_q(\mathfrak{g})$ as a "graded q-derivation", i.e. for $\psi \in A_q(\mathfrak{g})_{\gamma, \delta}$ and $\psi' \in A_q(\mathfrak{g})_{\gamma, \delta'}$ we have

$$e_j \cdot (\psi \psi') = (e_j \cdot \psi) \psi' + q^{\langle h_j, \delta \rangle} \psi (e_j \cdot \psi'). \tag{3.16}$$

Similarly,

$$(\psi \psi') \cdot f_j = \psi (\psi' \cdot f_j) + q^{\langle -h_j, \gamma' \rangle} (\psi \cdot f_j) \psi'. \tag{3.17}$$

Now, for $j \neq i$ we have $\langle h_j, \varpi_i \rangle = 0$ and $\langle h_j, s_i(\varpi_i) \rangle = \langle h_j, \varpi_i - \alpha_i \rangle = -a_{ij} \geq 0$. Thus, by (3.7), the left multiplication by e_j annihilates all the minors appearing in ψ_1, and as a consequence $e_j \cdot \psi_1 = 0$.

It remains to show that $e_i \cdot \psi_1 = 0$. To this end we observe first that $e_i \cdot \Delta_{u(\varpi_i), s_i(\varpi_i)} = \Delta_{u(\varpi_i), \varpi_i}$ for any $u \in W$. In fact,

$$e_i \cdot \Delta_{u(\varpi_i), s_i(\varpi_i)} = (e_i f_i) \cdot \Delta_{u(\varpi_i), \varpi_i} = (f_i e_i) \cdot \Delta_{u(\varpi_i), \varpi_i} + \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \cdot \Delta_{u(\varpi_i), \varpi_i}$$

$$= 0 + \frac{q^{\langle h_i, \varpi_i \rangle} - q^{\langle -h_i, \varpi_i \rangle}}{q - q^{-1}} \Delta_{u(\varpi_i), \varpi_i} = \Delta_{u(\varpi_i), \varpi_i}.$$

So, we can now calculate

$$e_i \cdot \psi_1 = (e_i \cdot \Delta_{s_i(\varpi_i), s_i(\varpi_i)}) \Delta_{\varpi_i, \varpi_i} - q^{-1}(e_i \cdot \Delta_{s_i(\varpi_i), \varpi_i}) \Delta_{\varpi_i, s_i(\varpi_i)}$$
$$+ q^{\langle h_i, s_i(\varpi) \rangle} \Delta_{s_i(\varpi_i), s_i(\varpi_i)}) (e_i \cdot \Delta_{\varpi_i, \varpi_i}) - q^{\langle h_i, \varpi_i \rangle - 1} \Delta_{s_i(\varpi_i), \varpi_i} (e_i \cdot \Delta_{\varpi_i, s_i(\varpi_i)})$$
$$= \Delta_{s_i(\varpi_i), \varpi_i} \Delta_{\varpi_i, \varpi_i} - q^0 \Delta_{s_i(\varpi_i), \varpi_i} \Delta_{\varpi_i, \varpi_i} = 0. \tag{3.18}$$

Finally, we have to show that $\psi_i \cdot f_j = 0$ for all $j \in I$. Again, for $j \neq i$ we see by (3.8) that the map $x \mapsto x \cdot f_j$ annihilates all the minors occuring in ψ_1. In order to see that $\psi_1 \cdot f_i = 0$, we note that by Lemma 3.6, we have $\Delta_{s_i(\varpi_i), \varpi_i} \Delta_{\varpi_i, s_i(\varpi_i)} = \Delta_{\varpi_i, s_i(\varpi_i)} \Delta_{s_i(\varpi_i), \varpi_i}$. Then we can proceed as in (3.18). Proposition 3.2 is proved.

4 The quantum coordinate rings $A_q(\mathfrak{b})$ and $A_q(\mathfrak{n})$

4.1 The quantum coordinate ring $A_q(\mathfrak{b})$

Let $U_q(\mathfrak{b})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i $(i \in I)$ and q^h $(h \in P^*)$. We have

$$\Delta(U_q(\mathfrak{b})) \subset U_q(\mathfrak{b}) \otimes U_q(\mathfrak{b}), \qquad S(U_q(\mathfrak{b})) \subseteq U_q(\mathfrak{b}), \tag{4.1}$$

hence $U_q(\mathfrak{b})$ is a Hopf subalgebra. Therefore, as in (2.10), $U_q(\mathfrak{b})^*$ has a multiplication dual to the comultiplication of $U_q(\mathfrak{b})$. Clearly, the map $\rho: U_q(\mathfrak{g})^* \to U_q(\mathfrak{b})^*$ given by restricting linear forms from $U_q(\mathfrak{g})$ to $U_q(\mathfrak{b})$ is an algebra homomorphism.

We define $A_q(\mathfrak{b}) := \rho(A_q(\mathfrak{g}))$. Let $Q_+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$. For $\gamma, \delta \in P$, let $A_q(\mathfrak{b})_{\gamma,\delta} = \rho(A_q(\mathfrak{g})_{\gamma,\delta})$. Since $U_q(\mathfrak{b}) \subseteq \bigoplus_{\alpha \in Q_+} U_q(\mathfrak{g})_\alpha$, we have by Lemma 2.2,

$$A_q(\mathfrak{b}) = \bigoplus_{\gamma - \delta \in Q_+} A_q(\mathfrak{b})_{\gamma,\delta}. \tag{4.2}$$

4.2 The quantum coordinate ring $A_q(\mathfrak{n})$

Recall that $U_q(\mathfrak{n})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by e_i $(i \in I)$. Because of (2.7), this is *not* a Hopf subalgebra. Nevertheless, we can endow $U_q(\mathfrak{n})^*$ with a multiplication as follows. Recall that every $y \in U_q(\mathfrak{b})$ can be written as a $\mathbb{Q}(q)$-linear combination of elements of the form xq^h with $x \in U_q(\mathfrak{n})$ and $h \in P^*$. Given $\psi \in U_q(\mathfrak{n})^*$, we define the linear form $\widetilde{\psi} \in U_q(\mathfrak{b})^*$ by

$$\widetilde{\psi}(xq^h) = \psi(x), \qquad (x \in U_q(\mathfrak{n}),\ h \in P^*). \tag{4.3}$$

Clearly, $\iota: \psi \mapsto \widetilde{\psi}$ is an injective linear map. Moreover, since $\Delta(xq^h) = \sum x_{(1)}q^h \otimes x_{(2)}q^h$, it follows immediately from (2.10) that

$$(\widetilde{\psi} \cdot \widetilde{\varphi})(xq^h) = (\widetilde{\psi} \cdot \widetilde{\varphi})(x), \qquad (\psi, \varphi \in U_q(\mathfrak{n})^*,\ x \in U_q(\mathfrak{n}),\ h \in P^*). \tag{4.4}$$

Therefore $\iota(U_q(\mathfrak{n})^*)$ is a subalgebra of $U_q(\mathfrak{b})^*$, and we can define

$$\psi \cdot \varphi = \iota^{-1}(\widetilde{\psi} \cdot \widetilde{\varphi}). \tag{4.5}$$

We have $U_q(\mathfrak{n}) = \bigoplus_{\alpha \in Q_+} U_q(\mathfrak{n})_\alpha$, where $U_q(\mathfrak{n})_\alpha = U_q(\mathfrak{n}) \cap U_q(\mathfrak{g})_\alpha$ is finite-dimensional for every $\alpha \in Q_+$. Let

$$A_q(\mathfrak{n}) = \bigoplus_{\alpha \in Q_+} \mathrm{Hom}_{\mathbb{Q}(q)}(U_q(\mathfrak{n})_\alpha, \mathbb{Q}(q)) = \bigoplus_{\alpha \in Q_+} A_q(\mathfrak{n})_\alpha \subset U_q(\mathfrak{n})^* \tag{4.6}$$

denote the graded dual of $U_q(\mathfrak{n})$. It is easy to see that $A_q(\mathfrak{n})$ is a subalgebra of $U_q(\mathfrak{n})^*$ for the multiplication defined in (4.5). Moreover, $\iota(A_q(\mathfrak{n})) \subset A_q(\mathfrak{b})$, and more precisely (4.3) shows that

$$\iota(A_q(\mathfrak{n})_\alpha) = A_q(\mathfrak{b})_{\alpha,0}, \quad (\alpha \in Q_+). \tag{4.7}$$

To summarize, $A_q(\mathfrak{n})$ can be identified with the subalgebra $\iota(A_q(\mathfrak{n})) = \bigoplus_{\alpha \in Q_+} A_q(\mathfrak{b})_{\alpha,0}$ of $A_q(\mathfrak{b})$.

4.3 The algebra isomorphism between $A_q(\mathfrak{n})$ and $U_q(\mathfrak{n})$

For $i \in I$, let $\delta_i \in \mathrm{End}_{\mathbb{Q}(q)}(U_q(\mathfrak{n}))$ be the q-derivation defined by $\delta_i(e_j) = \delta_{ij}$ and

$$\delta_i(xy) = \delta_i(x)y + q^{\langle h_i, \alpha \rangle} x \, \delta_i(y), \qquad (x \in U_q(\mathfrak{n})_\alpha, \ y \in U_q(\mathfrak{n})). \tag{4.8}$$

It is well known that there exists a unique nondegenerate symmetric bilinear form on $U_q(\mathfrak{n})$ such that $(1,1) = 1$ and

$$(\delta_i(x), y) = (x, e_i y), \qquad (x \in U_q(\mathfrak{n}), \ y \in U_q(\mathfrak{n}), \ i \in I). \tag{4.9}$$

Denote by ψ_x the linear form on $U_q(\mathfrak{n})$ given by $\psi_x(y) = (x, y)$. Then, the map $\Psi \colon x \mapsto \psi_x$ is an isomorphism of graded vector spaces from $U_q(\mathfrak{n})$ to $A_q(\mathfrak{n})$.

Proposition 4.1 *Let $A_q(\mathfrak{n})$ be endowed with the multiplication (4.5). Then Ψ is an isomorphism of algebras from $U_q(\mathfrak{n})$ to $A_q(\mathfrak{n})$.*

Proof — We need to show that

$$\widetilde{\psi}_{xy}(zq^h) = (\widetilde{\psi}_x \cdot \widetilde{\psi}_y)(zq^h), \quad (x, y, z \in U_q(\mathfrak{n}), \ h \in P^*). \tag{4.10}$$

By linearity, we can assume that $z = e_{i_1} \cdots e_{i_m}$ for some $i_1, \ldots, i_m \in I$. By definition,

$$\widetilde{\psi}_{xy}(zq^h) = (xy, z) = (\delta_{i_m} \cdots \delta_{i_1}(xy), 1). \tag{4.11}$$

Let us assume, without loss of generality, that $x \in U_q(\mathfrak{n})_\alpha$. It follows from (4.8) that

$$\delta_{i_m} \cdots \delta_{i_1}(xy) = \sum_K q^{\sigma(J,K)} \delta_{i_{j_s}} \cdots \delta_{i_{j_1}}(x) \, \delta_{i_{k_r}} \cdots \delta_{i_{k_1}}(y) \tag{4.12}$$

where the sum is over all subsets $K = \{k_1 < \cdots < k_r\}$ of $[1, m]$, $J = \{j_1 < \cdots < j_s\}$ is the complement of K in $[1, m]$, and

$$\sigma(J,K) = \left\langle h_{k_1}, \alpha - \sum_{j \in J, \, j < k_1} \alpha_j \right\rangle + \cdots + \left\langle h_{k_r}, \alpha - \sum_{j \in J, \, j < k_r} \alpha_j \right\rangle.$$

Moreover, a summand of the r.h.s. of (4.12) can give a nonzero contribution to (4.11) only if $\alpha = \sum_{j \in J} \alpha_j$. In this case we have

$$\sigma(J,K) = \left\langle h_{k_1}, \sum_{j \in J, \, j > k_1} \alpha_j \right\rangle + \cdots + \left\langle h_{k_r}, \sum_{j \in J, \, j > k_r} \alpha_j \right\rangle, \tag{4.13}$$

and

$$(xy, z) = \sum_K q^{\sigma(J,K)} (e_{i_{j_1}} \cdots e_{i_{j_s}}, x)(e_{i_{k_1}} \cdots e_{i_{k_r}}, y). \tag{4.14}$$

On the other hand, it is easy to deduce from (2.7) that

$$\Delta(z) = \sum_K q^{\sigma(J,K)} e_{i_{j_1}} \cdots e_{i_{j_s}} t_{i_{k_1}} \cdots t_{i_{k_r}} \otimes e_{i_{k_1}} \cdots e_{i_{k_r}}$$

where $\sigma(J,K)$ is again given by (4.13). It then follows from the definition of $\widetilde{\psi}_x$ and $\widetilde{\psi}_y$ that

$$(\widetilde{\psi}_x \cdot \widetilde{\psi}_y)(zq^h) = (\widetilde{\psi}_x \cdot \widetilde{\psi}_y)(z) = \sum_K q^{\sigma(J,K)} (e_{i_{j_1}} \cdots e_{i_{j_s}}, x)(e_{i_{k_1}} \cdots e_{i_{k_r}}, y) = \widetilde{\psi}_{xy}(zq^h).$$

$\square$

5 Determinantal identities for unipotent quantum minors

5.1 Principal quantum minors

Let $\lambda \in P_+$, and $u, v \in W$. The quantum minor $\Delta_{u(\lambda),v(\lambda)}$ is called *principal* when $u(\lambda) = v(\lambda)$. In this case we have by Lemma 2.2 (c) that $\Delta_{v(\lambda),v(\lambda)}(x) = 0$ if $x \notin U_q(\mathfrak{g})_0$. Therefore the restriction $\rho(\Delta_{v(\lambda),v(\lambda)}) \in A_q(\mathfrak{b})$ is given by

$$\rho(\Delta_{v(\lambda),v(\lambda)})(xq^h) = \varepsilon(x)q^{\langle h,v(\lambda)\rangle}, \qquad (x \in U_q(\mathfrak{n}),\ h \in P^*). \tag{5.1}$$

Define

$$\Delta^*_{v(\lambda),v(\lambda)} := \Delta_{v(\lambda),v(\lambda)} \circ S \in U_q(\mathfrak{g})^*, \tag{5.2}$$

where S is the antipode of $U_q(\mathfrak{g})$.

Lemma 5.1 *(a) The principal quantum minors $\rho(\Delta_{v(\lambda),v(\lambda)})$ ($v \in W$, $\lambda \in P_+$) are invertible in $A_q(\mathfrak{b})$, with inverse equal to $\rho(\Delta^*_{v(\lambda),v(\lambda)})$.*

(b) The principal quantum minors $\rho(\Delta_{v(\lambda),v(\lambda)})$ are q-central in $A_q(\mathfrak{b})$. More precisely, for $\psi \in A_q(\mathfrak{b})_{\gamma,\delta}$ we have

$$\rho(\Delta_{v(\lambda),v(\lambda)}) \cdot \psi = q^{(v(\lambda),\gamma-\delta)} \psi \cdot \rho(\Delta_{v(\lambda),v(\lambda)}).$$

Proof — (a) First, we note that $\rho(\Delta^*_{v(\lambda),v(\lambda)})$ belongs to $A_q(\mathfrak{b})$. Indeed, let $U_q(\mathfrak{b}_-)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by f_i ($i \in I$) and q^h ($h \in P^*$), and U_q^0 the subalgebra generated by q^h ($h \in P^*$). Since S is an anti-automorphism of $U_q(\mathfrak{g})$ which stabilizes $U_q(\mathfrak{b})$, $U_q(\mathfrak{b}_-)$, and U_q^0, it is clear that $\Delta^*_{v(\lambda),v(\lambda)}$ generates an integrable left (*resp.* right) submodule of $U_q(\mathfrak{g})^*$. So $\Delta^*_{v(\lambda),v(\lambda)} \in A_q(\mathfrak{g})$, and $\rho(\Delta^*_{v(\lambda),v(\lambda)}) \in A_q(\mathfrak{b})$. Now, it follows easily from the definition that

$$\Delta^*_{v(\lambda),v(\lambda)}(xq^h) = \varepsilon(x)q^{-\langle h,v(\lambda)\rangle}, \qquad (x \in U_q(\mathfrak{n}),\ h \in P^*),$$

so that

$$\left(\Delta_{v(\lambda),v(\lambda)} \cdot \Delta^*_{v(\lambda),v(\lambda)}\right)(xq^h) = \sum \Delta_{v(\lambda),v(\lambda)}(x_{(1)}q^h)\Delta^*_{v(\lambda),v(\lambda)}(x_{(2)}q^h) = \varepsilon(xq^h),$$

which proves (a).

(b) As for (a), it is enough to evaluate each side of the equation at a typical element xq^h of $U_q(\mathfrak{b})$. Since the equation relates two elements of $A_q(\mathfrak{b})_{\gamma+v(\lambda),\delta+v(\lambda)}$, we may assume that $x \in U_{\gamma-\delta}$. By linearity, we may further assume that $x = e_{i_1} \cdots e_{i_k}$, where $\alpha_{i_1} + \cdots + \alpha_{i_k} = \gamma - \delta$, without loss of generality. Using (2.7), we have

$$\left(\Delta_{v(\lambda),v(\lambda)} \cdot \psi\right)(xq^h) = \sum \Delta_{v(\lambda),v(\lambda)}(x_{(1)}q^h)\psi(x_{(2)}q^h) = \Delta_{v(\lambda),v(\lambda)}(t_{i_1}\cdots t_{i_k}q^h)\psi(xq^h),$$

because $\Delta_{v(\lambda),v(\lambda)}(x_{(1)}q^h) \neq 0$ only if $x_{(1)} \in U_q(\mathfrak{g})_0$. Hence

$$\left(\Delta_{v(\lambda),v(\lambda)} \cdot \psi\right)(xq^h) = q^{(v(\lambda),\gamma-\delta)}\Delta_{v(\lambda),v(\lambda)}(q^h)\psi(xq^h).$$

On the other hand it also follows from (2.7) that

$$\left(\psi \cdot \Delta_{v(\lambda),v(\lambda)}\right)(xq^h) = \sum \psi(x_{(1)}q^h)\Delta_{v(\lambda),v(\lambda)}(x_{(2)}q^h) = \psi(xq^h)\Delta_{v(\lambda),v(\lambda)}(q^h),$$

hence the result. □

5.2 Unipotent quantum minors

The quantum minor $\Delta_{u(\lambda),v(\lambda)}$ belongs to $A_q(\mathfrak{g})_{u(\lambda),v(\lambda)}$, hence its restriction $\rho(\Delta_{u(\lambda),v(\lambda)})$ belongs to $A_q(\mathfrak{b})_{u(\lambda),v(\lambda)}$. Therefore, if $v(\lambda) \neq 0$, it does not belong to $\iota(A_q(\mathfrak{n}))$. But a slight modification does, as we shall now see.

We define the *unipotent quantum minor* $D_{u(\lambda),v(\lambda)}$ by

$$D_{u(\lambda),v(\lambda)} = \rho\left(\Delta_{u(\lambda),v(\lambda)} \cdot \Delta^*_{v(\lambda),v(\lambda)}\right). \tag{5.3}$$

This is an element of $A_q(\mathfrak{b})_{u(\lambda)-v(\lambda),0} \subset \iota(U_q(\mathfrak{n})^*)$. In fact, the same calculation as in the proof of Lemma 5.1 shows that for $x \in U_q(\mathfrak{n})$ and $h \in P^*$, we have

$$D_{u(\lambda),v(\lambda)}(xq^h) = \Delta_{u(\lambda),v(\lambda)}(xq^h)\Delta_{v(\lambda),v(\lambda)}(q^{-h}) = \Delta_{u(\lambda),v(\lambda)}(x) = D_{u(\lambda),v(\lambda)}(x). \tag{5.4}$$

Thus, we can regard $D_{u(\lambda),v(\lambda)}$ as the restriction to $U_q(\mathfrak{n})$ of the quantum minor $\Delta_{u(\lambda),v(\lambda)}$. But we should be aware that this restriction is *not* an algebra homomorphism. For example, by Lemma 3.5 we have

$$\Delta_{u(\lambda),v(\lambda)} \cdot \Delta_{u(\mu),v(\mu)} = \Delta_{u(\mu),v(\mu)} \cdot \Delta_{u(\lambda),v(\lambda)}, \quad (u,v \in W, \ \lambda,\mu \in P_+).$$

The corresponding commutation relation for unipotent quantum minors is given by the following

Lemma 5.2 *For $u,v \in W$ and $\lambda,\mu \in P_+$ we have*

$$D_{u(\lambda),v(\lambda)} \cdot D_{u(\mu),v(\mu)} = q^{(v(\mu),u(\lambda))-(v(\lambda),u(\mu))}D_{u(\mu),v(\mu)} \cdot D_{u(\lambda),v(\lambda)}.$$

Proof — Let us write for short $\rho(\Delta_{u(\lambda),v(\lambda)}) = \Delta_{u(\lambda),v(\lambda)}$ and $\rho(\Delta^*_{v(\lambda),v(\lambda)}) = \Delta^*_{v(\lambda),v(\lambda)}$. We have, by Lemma 5.1,

$$
\begin{aligned}
D_{u(\lambda),v(\lambda)}D_{u(\mu),v(\mu)} &= \Delta_{u(\lambda),v(\lambda)}\Delta^*_{v(\lambda),v(\lambda)}\Delta_{u(\mu),v(\mu)}\Delta^*_{v(\mu),v(\mu)} \\
&= q^{(v(\lambda),v(\mu)-u(\mu))}\Delta_{u(\lambda),v(\lambda)}\Delta_{u(\mu),v(\mu)}\Delta^*_{v(\lambda),v(\lambda)}\Delta^*_{v(\mu),v(\mu)} \\
&= q^{(v(\lambda),v(\mu)-u(\mu))}\Delta_{u(\mu),v(\mu)}\Delta_{u(\lambda),v(\lambda)}\Delta^*_{v(\mu),v(\mu)}\Delta^*_{v(\lambda),v(\lambda)} \\
&= q^{(v(\mu),u(\lambda))-(v(\lambda),u(\mu))}\Delta_{u(\mu),v(\mu)}\Delta^*_{v(\mu),v(\mu)}\Delta_{u(\lambda),v(\lambda)}\Delta^*_{v(\lambda),v(\lambda)} \\
&= q^{(v(\mu),u(\lambda))-(v(\lambda),u(\mu))}D_{u(\mu),v(\mu)}D_{u(\lambda),v(\lambda)}
\end{aligned}
$$

$\square$

Similarly, Lemma 3.6 implies:

Lemma 5.3 *Suppose that for $u,v \in W$ and $i \in I$ we have $l(us_i) = l(u)+1$ and $l(vs_i) = l(v)+1$. Then*

$$D_{us_i(\varpi_i),v(\varpi_i)}D_{u(\varpi_i),vs_i(\varpi_i)} = q^{(vs_i(\varpi_i),us_i(\varpi_i))-(v(\varpi_i),u(\varpi_i))}D_{u(\varpi_i),vs_i(\varpi_i)}D_{us_i(\varpi_i),v(\varpi_i)}.$$

$\square$

We can also regard the quantum unipotent minors as linear forms on $U_q(\mathfrak{n})$ given by matrix coefficients of integrable representations. Indeed, using (3.11), we have

$$D_{u(\lambda),v(\lambda)}(x) = (m_{u(\lambda)}, xm_{v(\lambda)})_\lambda, \qquad (x \in U_q(\mathfrak{n})). \tag{5.5}$$

In particular, when $u = e$ we get the same quantum flag minors

$$D_{\lambda,v(\lambda)}(x) = (m_\lambda, xm_{v(\lambda)})_\lambda, \qquad (x \in U_q(\mathfrak{n})). \tag{5.6}$$

as in [**Ki**, §6.1] (up to a switch from $U_q(\mathfrak{n}_-)$ to $U_q(\mathfrak{n})$).

5.3 A family of identities for unipotent quantum minors

We are now in a position to deduce from Proposition 3.2 an algebraic identity satisfied by unipotent quantum minors. Later on, we will see that particular cases of this identity can be seen as quantum exchange relations in certain quantum cluster algebras.

As in §3.3, let us write $\gamma_i = \varpi_i + s_i(\varpi_i)$, so that

$$\Delta_{u(\gamma_i),v(\gamma_i)} = \prod_{j \neq i} \Delta_{u(\varpi_j),v(\varpi_j)}^{-a_{ji}}, \tag{5.7}$$

where, by Lemma 3.5, the order of the factors in the right-hand side is irrelevant.

Proposition 5.4 *Suppose that for* $u,v \in W$ *and* $i \in I$ *we have* $l(us_i) = l(u)+1$ *and* $l(vs_i) = l(v)+1$. *Then*

$$q^A D_{us_i(\varpi_i),vs_i(\varpi_i)} D_{u(\varpi_i),v(\varpi_i)} = q^{-1+B} D_{us_i(\varpi_i),v(\varpi_i)} D_{u(\varpi_i),vs_i(\varpi_i)} + D_{u(\gamma_i),v(\gamma_i)}$$

holds in $A_q(\mathfrak{n})$, *where*

$$A = (vs_i(\varpi_i),\, u(\varpi_i) - v(\varpi_i)), \qquad B = (v(\varpi_i),\, u(\varpi_i) - vs_i(\varpi_i)).$$

Proof — Again let us write for short $\rho(\Delta_{u(\lambda),v(\lambda)}) = \Delta_{u(\lambda),v(\lambda)}$ and $\rho(\Delta_{v(\lambda),v(\lambda)}^*) = \Delta_{v(\lambda),v(\lambda)}^*$. We apply the restriction homomorphism ρ to the equality of Proposition 3.2, and we multiply both sides from the right by

$$\Delta_{v(\varpi_i),v(\varpi_i)}^* \Delta_{v(s_i(\varpi_i)),v(s_i(\varpi_i))}^* = \Delta_{v(\gamma_i),v(\gamma_i)}^*.$$

Note that all these minors commute by Lemma 3.5. The result then follows directly from the definition of unipotent quantum minors, and from Lemma 5.1 (b).   □

It is sometimes useful to write the second summand of the right-hand side of Proposition 5.4 as a product. It is straightforward to deduce from (5.7) and Lemma 5.1 (b) that we have

$$D_{u(\gamma_i),v(\gamma_i)} = q^C \overrightarrow{\prod_{j \neq i}} \left(D_{u(\varpi_j),v(\varpi_j)} \right)^{-a_{ij}}, \tag{5.8}$$

where

$$C = \sum_{\substack{j<k \\ j \neq i \neq k}} a_{ij} a_{ik} (v(\varpi_j),\, u(\varpi_k) - v(\varpi_k)) + \sum_{j \neq i} \binom{-a_{ij}}{2} (v(\varpi_j),\, u(\varpi_j) - v(\varpi_j)). \tag{5.9}$$

5.4 A quantum T-system

Let $\mathbf{i} = (i_1, \ldots, i_r) \in I^r$ be such that $l(s_{i_1} \cdots s_{i_r}) = r$. We will now deduce from Proposition 5.4 a system of identities relating the unipotent quantum minors

$$D(k,l;j) := D_{s_{i_1} \cdots s_{i_k}(\varpi_j),\, s_{i_1} \cdots s_{i_l}(\varpi_j)}, \qquad (0 \leq k < l \leq r,\ j \in I). \tag{5.10}$$

Here, we use the convention that $D(0,l;j) = D_{\varpi_j,\, s_{i_1} \cdots s_{i_l}(\varpi_j)}$, a *quantum flag minor*. This system can be viewed as a q-analogue of a T-system, (see [**KNS**]). It will allow us to express every quantum minor $D(k,l;j)$ in terms of the flag minors $D(0,m;i)$.

Note that, because of (3.12), every quantum minor $D(k,l;j)$ is equal to a minor of the form $D(b,d;j)$ where $i_b = i_d = j$. When this is the case, we can simply write $D(b,d;j) = D(b,d)$. Note that in particular, $D(b,b) = 1$ for every b. By convention, we write $D(0,b) = D_{\varpi_{i_b}, \, s_{i_1} \cdots s_{i_b}(\varpi_{i_b})}$. We will also use the following shorthand notation:

$$b^-(j) \quad := \quad \max\left(\{s < b \mid i_s = j\} \cup \{0\}\right), \tag{5.11}$$

$$b^- \quad := \quad \max\left(\{s < b \mid i_s = i_b\} \cup \{0\}\right), \tag{5.12}$$

$$\mu(b,j) \quad := \quad s_{i_1} \cdots s_{i_b}(\varpi_j). \tag{5.13}$$

In (5.13) we understand that $\mu(0,j) = \varpi_j$. Clearly, we have $D(b,d;j) = D(b^-(j), d^-(j))$.

Proposition 5.5 *Let* $1 \le b < d \le r$ *be such that* $i_b = i_d = i$. *There holds*

$$q^A D(b,d) D(b^-,d^-) = q^{-1+B} D(b,d^-) D(b^-,d) + q^C \overrightarrow{\prod_{j \ne i}} D(b^-(j), d^-(j))^{-a_{ij}} \tag{5.14}$$

where

$$A = (\mu(d,i), \, \mu(b^-,i) - \mu(d^-,i)), \qquad B = (\mu(d^-,i), \, \mu(b^-,i) - \mu(d,i)),$$

and

$$C = \sum_{\substack{j<k \\ j \ne i \ne k}} a_{ij} a_{ik} (\mu(d,j), \, \mu(b,k) - \mu(d,k)) + \sum_{j \ne i} \binom{-a_{ij}}{2} (\mu(d,j), \, \mu(b,j) - \mu(d,j)).$$

Proof — This follows directly from Proposition 5.4, (5.8), and (5.9), by taking

$$u = s_{i_1} \cdots s_{i_{b-1}}, \qquad v = s_{i_1} \cdots s_{i_{d-1}}, \qquad i = i_b = i_d.$$

$\square$

6 Canonical bases

6.1 The canonical basis of $U_q(\mathfrak{n})$

We briefly review Lusztig's definition of a canonical basis of $U_q(\mathfrak{n})$.

Recall the scalar product $(\cdot, \cdot)$ on $U_q(\mathfrak{n})$ defined in §4.3. In [**Lu2**, Chapter 1], Lusztig defines a similar scalar product $(\cdot, \cdot)_L$, using the same q-derivation δ_i (denoted by $_i r$ in [**Lu2**, 1.2.13]) but with a different normalization $(e_i, e_i)_L = (1 - q^{-2})^{-1}$. It it easy to see that $(x,y) = 0$ if and only if $(x,y)_L = 0$, and if $x, y \in U_q(\mathfrak{n})_\beta$ then

$$(x,y)_L = (1 - q^{-2})^{-\deg \beta} (x,y), \tag{6.1}$$

where, for $\beta = \sum_i c_i \alpha_i$, we set $\deg \beta = \sum_i c_i$. This slight difference will not affect the definition of the canonical basis below, and we will always use $(\cdot, \cdot)$ instead of $(\cdot, \cdot)_L$.

Let $\mathbb{A} = \mathbb{Q}[q, q^{-1}]$. We introduce the $\mathbb{A}$-subalgebra $U_{\mathbb{A}}(\mathfrak{n})$ of $U_q(\mathfrak{n})$ generated by the divided powers $e_i^{(k)}$ ($i \in I$, $k \in \mathbb{N}$). We define a ring automorphism $x \mapsto \bar{x}$ of $U_q(\mathfrak{g})$ by

$$\bar{q} = q^{-1}, \quad \bar{e}_i = e_i, \quad \bar{f}_i = f_i, \qquad (i \in I). \tag{6.2}$$

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This restricts to a ring automorphism of $U_q(\mathfrak{n})$.

The canonical basis $\mathbf{B}$ is an $\mathbb{A}$-basis of $U_{\mathbb{A}}(\mathfrak{n})$ such that

$$\overline{b} = b, \qquad (b \in \mathbf{B}). \tag{6.3}$$

Moreover, for every $b, b' \in \mathbf{B}$ the scalar product $(b, b') \in \mathbb{Q}(q)$ has no pole at $q = \infty$, and

$$(b, b')|_{q=\infty} = \delta_{b,b'}. \tag{6.4}$$

By this we mean that

$$(b, b') = \frac{a_j q^j + \cdots a_1 q + a_0}{a'_k q^k + \cdots + a'_1 q + a'_0}, \qquad (a_i, a'_i \in \mathbb{Z}, \ a_j \neq 0, \ a'_k \neq 0),$$

with $j < k$ when $b \neq b'$, and $j = k, a_j = a'_k$ when $b = b'$. It is easy to see that if an $\mathbb{A}$-basis of $U_{\mathbb{A}}(\mathfrak{n})$ satisfies (6.3) and (6.4), then it is unique up to sign (see [**Lu2**, 14.2]). The existence of $\mathbf{B}$ is proved in [**Lu2**, Part 2], and a consistent choice of signs is provided. Of course, $\mathbf{B}$ is also a $\mathbb{Q}(q)$-basis of $U_q(\mathfrak{n})$.

6.2 The dual canonical basis of $U_q(\mathfrak{n})$

Let $\mathbf{B}^*$ be the basis of $U_q(\mathfrak{n})$ adjoint to $\mathbf{B}$ with respect to the scalar product $(\cdot, \cdot)$. We call it the *dual canonical basis* of $U_q(\mathfrak{n})$, since it can be identified via Ψ with the dual basis of $\mathbf{B}$ in $A_q(\mathfrak{n})$.

Note that $\mathbf{B}^*$ is not invariant under the bar automorphism $x \mapsto \overline{x}$. The property of $\mathbf{B}^*$ dual to (6.3) can be stated as follows. Let σ be the composition of the anti-automorphism $*$ and the bar involution, that is, σ is the ring *anti*-automorphism of $U_q(\mathfrak{n})$ such that

$$\sigma(q) = q^{-1}, \qquad \sigma(e_i) = e_i. \tag{6.5}$$

For $\beta \in Q_+$, define

$$N(\beta) := \frac{(\beta, \beta)}{2} - \deg \beta. \tag{6.6}$$

Then, if $b \in U_q(\mathfrak{n})_\beta$ belongs to $\mathbf{B}^*$, there holds

$$\sigma(b) = q^{N(\beta)} b, \tag{6.7}$$

(see [**Re**, **Ki**]).

6.3 Specialization at $q = 1$ of $U_q(\mathfrak{n})$ and $A_q(\mathfrak{n})$

Recall that $U_{\mathbb{A}}(\mathfrak{n})$ is the $\mathbb{A}$-submodule of $U_q(\mathfrak{n})$ spanned by the canonical basis $\mathbf{B}$. If we regard $\mathbb{C}$ as an $\mathbb{A}$-module via the homomorphism $q \mapsto 1$, we can define

$$U_1(\mathfrak{n}) := \mathbb{C} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\mathfrak{n}). \tag{6.8}$$

This is a $\mathbb{C}$-algebra isomorphic to the enveloping algebra $U(\mathfrak{n})$.

Similarly, let $A_{\mathbb{A}}(\mathfrak{n})$ be the $\mathbb{A}$-submodule of $A_q(\mathfrak{n})$ spanned by the basis $\Psi(\mathbf{B}^*)$. Define

$$A_1(\mathfrak{n}) := \mathbb{C} \otimes_{\mathbb{A}} A_{\mathbb{A}}(\mathfrak{n}). \tag{6.9}$$

This is a $\mathbb{C}$-algebra isomorphic to the graded dual $U(\mathfrak{n})^*_{\mathrm{gr}}$. This commutative ring can be identified with the coordinate ring $\mathbb{C}[N]$ of a pro-unipotent pro-group N with Lie algebra the completion $\widehat{\mathfrak{n}}$ of $\mathfrak{n}$ (see [**GLS6**]).

6.4 Global bases of $U_q(\mathfrak{n}_-)$

We shall also use Kashiwara's lower global basis $\mathbf{B}^{\text{low}}$ of $U_q(\mathfrak{n}_-)$, constructed in [**K1**]. It was proved by Grojnowski and Lusztig that $\varphi(\mathbf{B}) = \mathbf{B}^{\text{low}}$, where φ is the anti-automorphism of (2.1).

For $i \in I$, we introduce the q-derivations e_i' and $_ie'$ of $U_q(\mathfrak{n}_-)$, defined by $e_i'(f_j) = {_ie'}(f_j) = \delta_{ij}$ and, for homogeneous elements $x, y \in U_q(\mathfrak{n}_-)$,

$$e_i'(xy) = e_i'(x)y + q^{\langle h_i, \text{wt}(x) \rangle} x e_i'(y), \tag{6.10}$$

$$_ie'(xy) = q^{\langle h_i, \text{wt}(y) \rangle} {_ie'}(x)y + x {_ie'}(y). \tag{6.11}$$

Note that $_ie' = * \circ e_i' \circ *$. Let us denote by $(\cdot, \cdot)_K$ the Kashiwara scalar product on $U_q(\mathfrak{n}_-)$. It is the unique symmetric bilinear form such that $(1, 1)_K = 1$, and

$$(f_i x, y)_K = (x, e_i'(y))_K, \qquad (x \in U_q(\mathfrak{n}_-), \ y \in U_q(\mathfrak{n}_-), \ i \in I). \tag{6.12}$$

It also satisfies

$$(x f_i, y)_K = (x, {_ie'}(y))_K, \qquad (x \in U_q(\mathfrak{n}_-), \ y \in U_q(\mathfrak{n}_-), \ i \in I). \tag{6.13}$$

Let $\overline{\varphi}$ be the composition of φ and the bar involution, that is, $\overline{\varphi}$ is the ring *anti*-automorphism of $U_q(\mathfrak{g})$ such that

$$\overline{\varphi}(q) = q^{-1}, \quad \overline{\varphi}(e_i) = f_i, \quad \overline{\varphi}(f_i) = e_i, \quad \overline{\varphi}(q^h) = q^{-h}. \tag{6.14}$$

The following lemma expresses the compatibility between the scalar products and q-derivations on $U_q(\mathfrak{n})$ and $U_q(\mathfrak{n}_-)$.

Lemma 6.1 *(a) For $i \in I$, we have $_ie' \circ \overline{\varphi} = \overline{\varphi} \circ \delta_i$.*

(b) For $x, y \in U_q(\mathfrak{n})$ we have $\overline{(x, y)} = (\overline{\varphi}(x), \overline{\varphi}(y))_K$.

Proof — As $_ie' \circ \overline{\varphi}$ and $\overline{\varphi} \circ \delta_i$ are both linear, it is enough to prove that $_ie' \circ \overline{\varphi}(z) = \overline{\varphi} \circ \delta_i(z)$ for any homogeneous element z of $U_q(\mathfrak{n})$. We prove this by induction on the degree of z. If the degree is 1, then this follows easily from the definition of all these maps. Then assume that the degree of z is bigger than one. Then, without loss of generality, we can assume that $z = xy$ with degrees of x and y smaller than the degree of z. Now

$$_ie'(\overline{\varphi}(xy)) = {_ie'}(\overline{\varphi}(y)\overline{\varphi}(x)) = q^{-\langle h_i, \beta \rangle} {_ie'}(\overline{\varphi}(y)) \overline{\varphi}(x) + \overline{\varphi}(y) {_ie'}(\overline{\varphi}(x)).$$

By induction on the degrees of x and y we can assume that $_ie'(\varphi(x)) = \overline{\varphi}(\delta_i(x))$ and $_ie'(\overline{\varphi}(y)) = \overline{\varphi}(\delta_i(y))$, so that

$$_ie'(\overline{\varphi}(xy)) = \overline{\varphi}\left(q^{\langle h_i, \beta \rangle} x \delta_i(y) + \delta_i(x)y\right) = \overline{\varphi}(\delta_i(xy)),$$

which proves (a). Then

$$(\overline{\varphi}(e_i x), \overline{\varphi}(y))_K = (\overline{\varphi}(x) f_i, \overline{\varphi}(y))_K = (\overline{\varphi}(x), {_ie'}(\overline{\varphi}(y)))_K = (\overline{\varphi}(x), \overline{\varphi}(\delta_i(y)))_K.$$

By induction on the degrees of x and y we can assume that $(\overline{\varphi}(x), \overline{\varphi}(\delta_i(y)))_K = \overline{(x, \delta_i(y))} = \overline{(e_i x, y)}$, which proves (b). $\qquad \square$

Let $\mathbf{B}^{\text{up}}$ denote the upper global basis of $U_q(\mathfrak{n}_-)$. This is the basis adjoint to $\mathbf{B}^{\text{low}}$ with respect to $(\cdot, \cdot)_K$. By Lemma 6.1(b), we also have $\mathbf{B}^{\text{up}} = \overline{\varphi}(\mathbf{B}^*)$.

Let us denote by $\mathscr{B}(\infty)$ the crystal of $U_q(\mathfrak{n}_-)$, and by b_∞ its highest weight element. The elements of $\mathbf{B}^{\text{low}}$ (*resp.* $\mathbf{B}^{\text{up}}$) are denoted by $G^{\text{low}}(b)$ ($b \in \mathscr{B}(\infty)$) (*resp.* $G^{\text{up}}(b)$). As usual, for every $i \in I$, one denotes by $\widetilde{e}_i$ and $\widetilde{f}_i$ the Kashiwara crystal operators of $\mathscr{B}(\infty)$. We will need the following well known property of the upper global basis, (see [**K2**, Lemma 5.1.1], [**Ki**, Corollary 3.16]):

Lemma 6.2 *Let* $b \in \mathscr{B}(\infty)$, *and put* $k = \max\{j \in \mathbb{N} \mid (_ie')^j(G^{\text{up}}(b)) \neq 0\}$. *Then, denoting by* $(_ie')^{(k)}$ *the kth q-divided power of* $_ie'$, *we have* $(_ie')^{(k)}(G^{\text{up}}(b)) \in \mathbf{B}^{\text{up}}$. *More precisely,*

$$(_ie')^{(k)}(G^{\text{up}}(b)) = G^{\text{up}}((\widetilde{e}_i^*)^k b),$$

where $\widetilde{e}_i^*$ *is the crystal operator obtained from* $\widetilde{e}_i$ *by conjugating with the involution* $*$ *of (2.2).*

The integer $k = \max\{j \in \mathbb{N} \mid (e_i')^j(G^{\text{up}}(b)) \neq 0\} = \max\{j \in \mathbb{N} \mid (\widetilde{e}_i)^j(b) \neq 0\}$ is denoted by $\varepsilon_i(b)$. Similarly, the integer $k = \max\{j \in \mathbb{N} \mid (_ie')^j(G^{\text{up}}(b)) \neq 0\} = \max\{j \in \mathbb{N} \mid (\widetilde{e}_i^*)^j(b) \neq 0\}$ is denoted by $\varepsilon_i^*(b)$.

6.5 Unipotent quantum minors belong to $\mathbf{B}^*$

Using the isomorphism $\Psi \colon U_q(\mathfrak{n}) \to A_q(\mathfrak{n})$ of 4.3, we can regard the unipotent quantum minors $D_{u(\lambda),v(\lambda)}$ as elements of $U_q(n)$. More precisely, let $d_{u(\lambda),v(\lambda)} = \Psi^{-1}(D_{u(\lambda),v(\lambda)})$ be the element of $U_q(\mathfrak{n})$ such that

$$D_{u(\lambda),v(\lambda)}(x) = (d_{u(\lambda),v(\lambda)}, x), \qquad (x \in U_q(\mathfrak{n})). \tag{6.15}$$

By a slight abuse, we shall also call $d_{u(\lambda),v(\lambda)}$ a unipotent quantum minor. In this section we show:

Proposition 6.3 *For every* $\lambda \in P_+$ *and* $u, v \in W$ *such that* $u(\lambda) - v(\lambda) \in Q_+$, *the unipotent quantum minor* $d_{u(\lambda),v(\lambda)}$ *belongs to* $\mathbf{B}^*$. *More precisely, writing* $u = s_{i_{l(u)}} \cdots s_{i_1}$, $v = s_{j_{l(v)}} \cdots s_{j_1}$ *and defining* b_k *and* c_l *as in (3.4), we have*

$$d_{u(\lambda),v(\lambda)} = \overline{\varphi}\left(G^{\text{up}}\left((\widetilde{e}_{i_{l(u)}}^*)^{b_{l(u)}} \cdots (\widetilde{e}_{i_1}^*)^{b_1} \widetilde{f}_{j_{l(v)}}^{c_{l(v)}} \cdots \widetilde{f}_{j_1}^{c_1} b_\infty\right)\right).$$

Proof — We proceed in two steps, and first consider the case when $u = e$ is the unit of W, that is, the case of unipotent quantum flag minors $d_{\lambda,v(\lambda)}$. By (5.6), we have for $x \in U_q(\mathfrak{n})$,

$$(d_{\lambda,v(\lambda)}, x) = (m_\lambda, x m_{v(\lambda)})_\lambda = (\varphi(x) m_\lambda, m_{v(\lambda)})_\lambda.$$

It is well known that the extremal weight vectors $m_{v(\lambda)}$ belong to Kashiwara's lower global basis of $V(\lambda)$, and also to the upper global basis. More precisely, we have

$$m_{v(\lambda)} = G^{\text{low}}\left(\widetilde{f}_{j_{l(v)}}^{c_{l(v)}} \cdots \widetilde{f}_{j_1}^{c_1} b_\lambda\right) = G^{\text{up}}\left(\widetilde{f}_{j_{l(v)}}^{c_{l(v)}} \cdots \widetilde{f}_{j_1}^{c_1} b_\lambda\right),$$

where b_λ is the highest weight element of the crystal $\mathscr{B}(\lambda)$ of $V(\lambda)$. Hence we have

$$(d_{\lambda,v(\lambda)}, x) = \left(\varphi(x) m_\lambda, G^{\text{up}}\left(\widetilde{f}_{j_{l(v)}}^{c_{l(v)}} \cdots \widetilde{f}_{j_1}^{c_1} b_\lambda\right)\right)_\lambda. \tag{6.16}$$

It follows from (6.16) that $(d_{\lambda,v(\lambda)}, x) = 1$ if $\varphi(x) = G^{\text{low}}\left(\widetilde{f}_{j_{l(v)}}^{c_{l(v)}} \cdots \widetilde{f}_{j_1}^{c_1} b_\infty\right)$, and $(d_{\lambda,v(\lambda)}, x) = 0$ if $\varphi(x)$ is any other element of $\mathbf{B}^{\text{low}}$. Therefore, $d_{\lambda,v(\lambda)}$ is the element of $\mathbf{B}^*$ given by

$$d_{\lambda,v(\lambda)} = \overline{\varphi}\left(G^{\text{up}}\left(\widetilde{f}_{j_{l(v)}}^{c_{l(v)}} \cdots \widetilde{f}_{j_1}^{c_1} b_\infty\right)\right).$$

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Now let us consider the general case when $u = s_{i_{l(u)}} \cdots s_{i_1}$ is non trivial. For $k = 1, \ldots, l(u)$, write $u_k = s_{i_k} \cdots s_{i_1}$. Using Lemma 3.1, we have for $x \in U_q(\mathfrak{n})$,

$$D_{u_k(\lambda), v(\lambda)}(x) = \Delta_{u_k(\lambda), v(\lambda)}(x) = \left(\Delta_{u_{k-1}(\lambda), v(\lambda)} \cdot e_{i_k}^{(b_k)} \right)(x)$$

where $b_k = (\alpha_{i_k}, u_{k-1}(\lambda)) = \max\{j \mid \Delta_{u_{k-1}(\lambda), v(\lambda)} \cdot e_{i_k}^{(j)} \neq 0\}$. Now we can also write

$$D_{u_k(\lambda), v(\lambda)}(x) = \left(d_{u_{k-1}(\lambda), v(\lambda)}, e_{i_k}^{(b_k)} x \right) = \left((\delta_{i_k})^{(b_k)} d_{u_{k-1}(\lambda), v(\lambda)}, x \right),$$

hence

$$d_{u_k(\lambda), v(\lambda)} = (\delta_{i_k})^{(b_k)} d_{u_{k-1}(\lambda), v(\lambda)},$$

where $(\delta_i)^{(b)}$ means the bth q-divided power of the q-derivation δ_i. Since $u_k(\lambda) - v(\lambda) \in Q_+$, the restriction $\rho(\Delta_{u_k(\lambda), v(\lambda)})$ is nonzero, hence $b_k = \max\{j \mid (\delta_{i_k})^{(b)} d_{u_{k-1}(\lambda), v(\lambda)} \neq 0\}$. Applying $\overline{\varphi}$ and assuming by induction on k that

$$d_{u_{k-1}(\lambda), v(\lambda)} = \overline{\varphi} \left(G^{\mathrm{up}} \left((\widetilde{e}_{i_{k-1}}^{*})^{b_{k-1}} \cdots (\widetilde{e}_{i_1}^{*})^{b_1} \widetilde{f}_{j_{l(v)}}^{c_{l(v)}} \cdots \widetilde{f}_{j_1}^{c_1} b_\infty \right) \right)$$

we get by Lemma 6.1(a) that

$$\overline{\varphi}(d_{u_k(\lambda), v(\lambda)}) = (_{i_k}e')^{(b_k)} \left(G^{\mathrm{up}} \left((\widetilde{e}_{i_{k-1}}^{*})^{b_{k-1}} \cdots (\widetilde{e}_{i_1}^{*})^{b_1} \widetilde{f}_{j_{l(v)}}^{c_{l(v)}} \cdots \widetilde{f}_{j_1}^{c_1} b_\infty \right) \right)$$

and

$$b_k = \varepsilon_{i_k}^{*} \left((\widetilde{e}_{i_{k-1}}^{*})^{b_{k-1}} \cdots (\widetilde{e}_{i_1}^{*})^{b_1} \widetilde{f}_{j_{l(v)}}^{c_{l(v)}} \cdots \widetilde{f}_{j_1}^{c_1} b_\infty \right).$$

Thus, applying Lemma 6.2, we get that

$$\overline{\varphi}(d_{u_k(\lambda), v(\lambda)}) = G^{\mathrm{up}} \left((\widetilde{e}_{i_k}^{*})^{b_k} \cdots (\widetilde{e}_{i_1}^{*})^{b_1} \widetilde{f}_{j_{l(v)}}^{c_{l(v)}} \cdots \widetilde{f}_{j_1}^{c_1} b_\infty \right),$$

and the statement follows by induction on k. $\qquad\qquad \square$

7 Quantum unipotent subgroups

In this section we provide a quantum version of the coordinate ring $\mathbb{C}[N(w)]$ studied in [**GLS6**], following [**Lu2**, **Sa**, **Ki**].

7.1 The quantum enveloping algebra $U_q(\mathfrak{n}(w))$

We fix $w \in W$, and we denote by Δ_w^+ the subset of positive roots α of $\mathfrak{g}$ such that $w(\alpha)$ is a negative root. This gives rise to a finite-dimensional Lie subalgebra

$$\mathfrak{n}(w) := \bigoplus_{\alpha \in \Delta_w^+} \mathfrak{n}_\alpha$$

of $\mathfrak{n}$, of dimension $l(w)$. The graded dual $U(\mathfrak{n}(w))_{\mathrm{gr}}^*$ can be identified with the coordinate ring $\mathbb{C}[N(w)]$ of a unipotent subgroup $N(w)$ of the Kac-Moody group G with $\mathrm{Lie}(N(w)) = \mathfrak{n}(w)$. (For more details, see [**GLS6**].)

In order to define a q-analogue of $U(\mathfrak{n}(w))$, one introduces Lusztig's braid group operation on $U_q(\mathfrak{g})$ [**Lu2**]. For $i \in I$, Lusztig has proved the existence of a $\mathbb{Q}(q)$-algebra automorphism T_i of $U_q(\mathfrak{g})$ satisfying

$$
\begin{align}
T_i(q^h) &= q^{s_i(h)}, \tag{7.1}\\
T_i(e_i) &= -t_i^{-1}f_i, \tag{7.2}\\
T_i(f_i) &= -e_i t_i, \tag{7.3}\\
T_i(e_j) &= \sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q^{-r} e_i^{(r)} e_j e_i^{(s)} \qquad (j \neq i), \tag{7.4}\\
T_i(f_j) &= \sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q^r f_i^{(s)} f_j f_i^{(r)} \qquad (j \neq i). \tag{7.5}
\end{align}
$$

(This automorphism is denoted by $T'_{i,-1}$ in [**Lu2**].) For a fixed reduced decomposition $w = s_{i_r}\cdots s_{i_1}$, let us set, as in (3.3),

$$
\beta_k = s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k}), \qquad (1 \le k \le r). \tag{7.6}
$$

Then $\Delta_w^+ = \{\beta_1,\ldots,\beta_r\}$. We define following Lusztig, the corresponding quantum root vectors:

$$
E(\beta_k) := T_{i_1}\cdots T_{i_{k-1}}(e_{i_k}), \qquad (1 \le k \le r). \tag{7.7}
$$

It is known that $E(\beta_k) \in U_q(\mathfrak{n})_{\beta_k}$. For $\mathbf{a} = (a_1,\ldots,a_r) \in \mathbb{N}^r$, set

$$
E(\mathbf{a}) := E(\beta_1)^{(a_1)}\cdots E(\beta_r)^{(a_r)}, \tag{7.8}
$$

where $E(\beta_k)^{(a_k)}$ denotes the a_kth q-divided power of $E(\beta_k)$. Lusztig has shown that the subspace of $U_q(\mathfrak{n})$ spanned by $\{E(\mathbf{a}) \mid \mathbf{a} \in \mathbb{N}^r\}$ is independent of the choice of the reduced word $\mathbf{i} = (i_r,\ldots,i_1)$ for w. We denote it by $U_q(\mathfrak{n}(w))$. Moreover,

$$
\mathscr{P}_\mathbf{i} := \{E(\mathbf{a}) \mid \mathbf{a} \in \mathbb{N}^r\} \tag{7.9}
$$

is a basis of $U_q(\mathfrak{n}(w))$, which we call the *PBW-basis* attached to $\mathbf{i}$.

In fact, $U_q(\mathfrak{n}(w))$ is even a subalgebra of $U_q(\mathfrak{n})$. This follows from a formula due to Levendorskii-Soibelman (see [**Ki**, 4.3.3]).

7.2 The quantum coordinate ring $A_q(\mathfrak{n}(w))$

Using the algebra isomorphism $\Psi \colon U_q(\mathfrak{n}) \to A_q(\mathfrak{n})$ of 4.3, we can define $A_q(\mathfrak{n}(w)) := \Psi(U_q(\mathfrak{n}(w)))$. This is a subalgebra of $A_q(\mathfrak{n})$.

Lusztig [**Lu2**, 38.2.3] has shown that $\mathscr{P}_\mathbf{i}$ is orthogonal, that is $(E(\mathbf{a}),E(\mathbf{b})) = 0$ if $\mathbf{a} \neq \mathbf{b}$. Moreover

$$
(E(\beta_k),E(\beta_k)) = (1-q^{-2})^{\deg\beta_k-1}, \qquad (1 \le k \le r), \tag{7.10}
$$

and

$$
(E(\mathbf{a}),E(\mathbf{a})) = \prod_{k=1}^r \frac{(E(\beta_k),E(\beta_k))^{a_k}}{\{a_k\}!}, \tag{7.11}
$$

where by definition

$$
\{a\}! = \prod_{j=1}^a \frac{1-q^{-2j}}{1-q^{-2}}. \tag{7.12}
$$

Denote by $\mathscr{P}_{\mathbf{i}}^*$ the basis

$$E^*(\mathbf{a}) = \frac{1}{(E(\mathbf{a}), E(\mathbf{a}))}\, E(\mathbf{a}), \qquad (\mathbf{a} \in \mathbb{N}^r) \tag{7.13}$$

of $U_q(\mathfrak{n}(w))$ adjoint to $\mathscr{P}_{\mathbf{i}}$. We call $\mathscr{P}_{\mathbf{i}}^*$ the *dual PBW-basis* of $U_q(\mathfrak{n}(w))$ since it can be identified via Ψ with the basis of $A_q(\mathfrak{n}(w))$ dual to $\mathscr{P}_{\mathbf{i}}$. In particular we have the dual PBW generators:

$$E^*(\beta_k) = (1 - q^{-2})^{-\deg \beta_k + 1} E(\beta_k), \qquad (1 \le k \le r). \tag{7.14}$$

7.3 Action of T_i on unipotent quantum minors

Proposition 7.1 *Let $\lambda \in P^+$, and $u, v \in W$ be such that $u(\lambda) - v(\lambda) \in Q_+$, and consider the unipotent quantum minor $d_{u(\lambda), v(\lambda)}$. Suppose that $l(s_i u) = l(u) + 1$, and $l(s_i v) = l(v) + 1$. Then*

$$T_i\left(d_{u(\lambda), v(\lambda)}\right) = (1 - q^{-2})^{(\alpha_i, v(\lambda) - u(\lambda))}\, d_{s_i u(\lambda), s_i v(\lambda)}.$$

The proof will use Proposition 6.3 and the following lemmas.

Lemma 7.2 *We have $T_i \circ \overline{\varphi} = \overline{\varphi} \circ T_i$.*

Proof — This follows immediately from the definitions of $\overline{\varphi}$ and of T_i. $\square$

The next lemma is a restatement of a result of Kimura [**Ki**, Theorem 4.20], based on previous results of Saito [**Sa**] and Lusztig [**Lu3**]. Note that our T_i is denoted by T_i^{-1} in [**Ki**].

Lemma 7.3 *Let $b \in \mathscr{B}(\infty)$ be such that $\varepsilon_i(b) = 0$. Then*

$$T_i(G^{\mathrm{up}}(b)) = (1 - q^2)^{(\alpha_i, \mathrm{wt}(b))}\, G^{\mathrm{up}}\left(\widetilde{f}_i^{\;\varphi_i^*(b)}(\widetilde{e}_i^*)^{\varepsilon_i^*(b)}b\right),$$

where $\varphi_i^*(b) := \varepsilon_i^*(b) + (\alpha_i, \mathrm{wt}(b))$.

Proof of Proposition 7.1 — By Lemma 7.2 and Proposition 6.3, we have

$$\overline{\varphi}\left(T_i(d_{u(\lambda), v(\lambda)})\right) = T_i\left(G^{\mathrm{up}}\left((\widetilde{e}_{i_{l(u)}}^*)^{b_{l(u)}} \cdots (\widetilde{e}_{i_1}^*)^{b_1} \widetilde{f}_{j_{l(v)}}^{\;c_{l(v)}} \cdots \widetilde{f}_{j_1}^{\;c_1} b_\infty\right)\right).$$

Let us write for short $b := (\widetilde{e}_{i_{l(u)}}^*)^{b_{l(u)}} \cdots (\widetilde{e}_{i_1}^*)^{b_1} \widetilde{f}_{j_{l(v)}}^{\;c_{l(v)}} \cdots \widetilde{f}_{j_1}^{\;c_1} b_\infty \in \mathscr{B}(\infty)$. Then

$$\varepsilon_i(b) = \max\{s \mid e_i^s \cdot \Delta_{u(\lambda), v(\lambda)} \ne 0\}.$$

The assumption $l(s_i v) = l(v) + 1$ implies that $(\alpha_i, v(\lambda)) \ge 0$, so by (3.7), we have $\varepsilon_i(b) = 0$, and we are in a position to apply Lemma 7.3. Because of the assumption $l(s_i u) = l(u) + 1$ we have

$$\varepsilon_i^*(b) = \max\{s \mid \Delta_{u(\lambda), v(\lambda)} \cdot e_i^s \ne 0\} = (\alpha_i, u(\lambda)),$$

and

$$\varphi_i^*(b) = \varepsilon_i^*(b) + (\alpha_i, \mathrm{wt}(b)) = (\alpha_i, u(\lambda)) + (\alpha_i, v(\lambda) - u(\lambda)) = (\alpha_i, v(\lambda)).$$

Thus, again by Proposition 6.3, we have

$$
\begin{aligned}
T_i(d_{u(\lambda),v(\lambda)}) &= \overline{\varphi}\left(T_i(G^{\mathrm{up}}b)\right) \\
&= \overline{\varphi}\left((1-q^2)^{(\alpha_i,v(\lambda)-u(\lambda))}G^{\mathrm{up}}\left(\widetilde{f}_i^{\,\varphi_i^*(b)}(\widetilde{e}_i^*)^{\varepsilon_i^*(b)}b\right)\right) \\
&= (1-q^{-2})^{(\alpha_i,v(\lambda)-u(\lambda))}\overline{\varphi}\left(G^{\mathrm{up}}\left((\widetilde{e}_i^*)^{\varepsilon_i^*(b)}(\widetilde{e}_{i_{l(u)}}^*)^{b_{l(u)}}\cdots(\widetilde{e}_{i_1}^*)^{b_1}\widetilde{f}_i^{\,\varphi_i^*(b)}\widetilde{f}_{j_{l(v)}}^{\,c_{l(v)}}\cdots\widetilde{f}_{j_1}^{\,c_1}b_\infty\right)\right) \\
&= (1-q^{-2})^{(\alpha_i,v(\lambda)-u(\lambda))}d_{s_iu(\lambda),s_iv(\lambda)}.
\end{aligned}
$$

$\square$

7.4 Dual PBW generators are unipotent quantum minors

Recall from (7.6) the definition of the roots β_k.

Proposition 7.4 *For $k = 1, \ldots, r$, we have*

$$
E^*(\beta_k) = d_{s_{i_1}\cdots s_{i_{k-1}}(\varpi_{i_k}),\, s_{i_1}\cdots s_{i_k}(\varpi_{i_k})}.
$$

Proof — We have $e_{i_k} = d_{\varpi_{i_k},\, s_{i_k}(\varpi_{i_k})}$, hence

$$
E(\beta_k) = T_{i_1}\cdots T_{i_{k-1}}\left(d_{\varpi_{i_k},\, s_{i_k}(\varpi_{i_k})}\right).
$$

Applying $k-1$ times Proposition 7.1, we get

$$
E(\beta_k) = (1-q^{-2})^N d_{s_{i_1}\cdots s_{i_{k-1}}(\varpi_{i_k}),\, s_{i_1}\cdots s_{i_k}(\varpi_{i_k})},
$$

where

$$
N = -(\alpha_{i_{k-1}},\alpha_{i_k}) - (\alpha_{i_{k-2}}, s_{i_{k-1}}(\alpha_{i_k})) - \cdots - (\alpha_{i_1}, s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k})).
$$

Now,

$$
\begin{aligned}
\beta_k &= s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k}) \\
&= \alpha_{i_k} + \sum_{n=1}^{k-1}\left(s_{i_n}\cdots s_{i_{k-1}}(\alpha_{i_k}) - s_{i_{n+1}}\cdots s_{i_{k-1}}(\alpha_{i_k})\right) \\
&= \alpha_{i_k} - \sum_{n=1}^{k-1}(\alpha_{i_n}, s_{i_{n+1}}\cdots s_{i_{k-1}}(\alpha_{i_k}))\alpha_{i_n},
\end{aligned}
$$

so that $\deg\beta_k = 1 + N$. Hence,

$$
d_{s_{i_1}\cdots s_{i_{k-1}}(\varpi_{i_k}),\, s_{i_1}\cdots s_{i_k}(\varpi_{i_k})} = (1-q^{-2})^{1-\deg\beta_k}E(\beta_k) = E^*(\beta_k).
$$

$\square$

7.5 The skew field $F_q(\mathfrak{n}(w))$

It is well known that $A_q(\mathfrak{n}(w))$ is an Ore domain. This follows for example from the fact that it has polynomial growth (see [**BZ2**, Appendix]). Hence $A_q(\mathfrak{n}(w))$ embeds in its skew field of fractions, which we shall denote by $F_q(\mathfrak{n}(w))$.

Recall the shorthand notation of §5.4 for unipotent quantum minors.

Proposition 7.5 *Let $0 \leq b < d \leq r$ be such that $i_b = i_d$. Then the unipotent quantum minor $D(b,d)$ belongs to $F_q(\mathfrak{n}(w))$. In particular, the quantum flag minors $D(0,d) = D_{\varpi_{i_d}, \, s_{i_1} \cdots s_{i_d}(\varpi_{i_d})}$ belong to $F_q(\mathfrak{n}(w))$.*

Proof — If $b = d^-$, by Proposition 7.4, we have $D(d^-, d) = \Psi(E^*(\beta_d))$, so

$$D(d^-, d) \in A_q(\mathfrak{n}(w)) \subset F_q(\mathfrak{n}(w)).$$

Recall the determinantal identity (5.14). Arguing as in [**GLS6**, Corollary 13.3], we can order the set of minors $D(b,d)$ $(0 \leq b < d \leq r)$ so that (i) the minors $D(d^-, d)$ are the smallest elements, and (ii) the minor $D(b^-, d)$ is strictly bigger than all the other minors occuring in (5.14). This allows to express, recursively $D(b^-, d)$ as a rational expression in the minors $D(c^-, c)$ $(1 \leq c \leq r)$, and shows that $D(b^-, d) \in F_q(\mathfrak{n}(w))$. □

We will show later (see Corollary 12.4) that, in fact, all quantum minors $D(b,d)$ are *polynomials* in the dual PBW-generators $D(c^-, c)$. Hence, they belong to $A_q(\mathfrak{n}(w))$.

7.6 Specialization at $q = 1$ of $A_q(\mathfrak{n}(w))$

Let $A_{\mathbb{A}}(\mathfrak{n}(w))$ denote the free $\mathbb{A}$-submodule of $A_q(\mathfrak{n}(w))$ with basis $\Psi(\mathscr{P}_{\mathbf{i}}^*)$. This integral form of $A_q(\mathfrak{n}(w))$ is an $\mathbb{A}$-algebra, independent of the choice of the reduced word $\mathbf{i}$. Moreover, if we regard $\mathbb{C}$ as an $\mathbb{A}$-module via the homomorphism $q \mapsto 1$, we can define

$$A_1(\mathfrak{n}(w)) := \mathbb{C} \otimes_{\mathbb{A}} A_{\mathbb{A}}(\mathfrak{n}(w)). \tag{7.15}$$

This is a $\mathbb{C}$-algebra isomorphic to the coordinate ring $\mathbb{C}[N(w)]$, (see [**Ki**, Theorem 4.39]). In particular, if $D_{u(\lambda), v(\lambda)}$ is a unipotent quantum minor in $A_q(\mathfrak{n}(w))$, the element $1 \otimes D_{u(\lambda), v(\lambda)}$ can be identified with the corresponding classical minor in $\mathbb{C}[N(w)]$.

8 Quantum cluster algebras

In this section we recall, following Berenstein and Zelevinsky [**BZ2**], the definition of a quantum cluster algebra.

8.1 Based quantum tori

Let $L = (\lambda_{ij})$ be a skew-symmetric $r \times r$-matrix with integer entries. The *based quantum torus* $\mathscr{T}(L)$ is the $\mathbb{Z}[q^{\pm 1/2}]$-algebra generated by symbols $X_1, \ldots, X_r, X_1^{-1}, \ldots, X_r^{-1}$ submitted to the relations

$$X_i X_i^{-1} = X_i^{-1} X_i = 1, \qquad X_i X_j = q^{\lambda_{ij}} X_j X_i, \qquad (1 \leq i, j \leq r). \tag{8.1}$$

For $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$, set

$$X^{\mathbf{a}} := q^{\frac{1}{2} \sum_{i>j} a_i a_j \lambda_{ij}} X_1^{a_1} \cdots X_r^{a_r}. \tag{8.2}$$

Then $\{X^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{Z}^r\}$ is a $\mathbb{Z}[q^{\pm 1/2}]$-basis of $\mathscr{T}(L)$, and we have for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^r$,

$$X^{\mathbf{a}} X^{\mathbf{b}} = q^{\frac{1}{2} \sum_{i>j} (a_i b_j - b_i a_j) \lambda_{ij}} X^{\mathbf{a}+\mathbf{b}} = q^{\sum_{i>j} (a_i b_j - b_i a_j) \lambda_{ij}} X^{\mathbf{b}} X^{\mathbf{a}}. \tag{8.3}$$

Since $\mathscr{T}(L)$ is an Ore domain, it can be embedded in its skew field of fractions $\mathscr{F}$.

8.2 Quantum seeds

Fix a positive integer $n < r$. Let $\widetilde{B} = (b_{ij})$ be an $r \times (r-n)$-matrix with integer coefficients. The submatrix B consisting of the first $r - n$ rows of $\widetilde{B}$ is called the *principal part* of $\widetilde{B}$. We will require B to be skew-symmetric. We call $\widetilde{B}$ an *exchange matrix*. We say that the pair $(L, \widetilde{B})$ is *compatible* if we have

$$\sum_{k=1}^r b_{kj} \lambda_{ki} = \delta_{ij} d, \qquad (1 \le j \le r-n, \ 1 \le i \le r) \tag{8.4}$$

for some positive integer d.

If $(L, \widetilde{B})$ is compatible, the datum $\mathscr{S} = ((X_1, \ldots, X_r), L, \widetilde{B})$ is called a *quantum seed* in $\mathscr{F}$. The set $\{X_1, \ldots, X_r\}$ is called the *cluster* of $\mathscr{S}$, and its elements the *cluster variables*. The cluster variables $X_{r-n+1}, \ldots, X_r$ are called *frozen variables*, since they will not be affected by the operation of mutation to be defined now. The elements $X^{\mathbf{a}}$ with $\mathbf{a} \in \mathbb{N}^r$ are called *quantum cluster monomials*.

8.3 Mutations

For $k = 1, \ldots, r - n$, we define the *mutation* $\mu_k(L, \widetilde{B})$ of a compatible pair $(L, \widetilde{B})$. Let E be the $r \times r$-matrix with entries

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \ne k, \\ -1 & \text{if } i = j = k, \\ \max(0, -b_{ik}) & \text{if } i \ne j = k. \end{cases} \tag{8.5}$$

Let F be the $(r-n) \times (r-n)$-matrix with entries

$$f_{ij} = \begin{cases} \delta_{ij} & \text{if } i \ne k, \\ -1 & \text{if } i = j = k, \\ \max(0, b_{kj}) & \text{if } i = k \ne j. \end{cases} \tag{8.6}$$

Then $\mu_k(L, \widetilde{B}) = (\mu_k(L), \mu_k(\widetilde{B}))$ where

$$\mu_k(L) := E^T L E, \qquad \mu_k(\widetilde{B}) := E \widetilde{B} F. \tag{8.7}$$

Note that the mutation $\mu_k(\widetilde{B})$ of the exchange matrix is a reformulation of the classical one defined in [**FZ2**]. It is easy to check that $\mu_k(L, \widetilde{B})$ is again a compatible pair, with the same integer d as in (8.4). Define $\mathbf{a}' = (a'_1, \ldots, a'_r)$ and $\mathbf{a}'' = (a''_1, \ldots, a''_r)$ by

$$a'_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, b_{ik}) & \text{if } i \ne k, \end{cases} \qquad a''_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, -b_{ik}) & \text{if } i \ne k. \end{cases} \tag{8.8}$$

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One then defines

$$\mu_k(X_i) = \begin{cases} X^{\mathbf{a}'} + X^{\mathbf{a}''}, & \text{if } i = k, \\ X_i & \text{if } i \neq k. \end{cases} \tag{8.9}$$

Berenstein and Zelevinsky show that the elements $X_i' := \mu_k(X_i)$ satisfy

$$X_i' X_j' = q^{\lambda_{ij}'} X_j' X_i', \qquad (1 \leq i, j \leq r), \tag{8.10}$$

where $\mu_k(L) = (\lambda_{ij}')$. Moreover they form a *free generating set* of $\mathscr{F}$, that is, one can write $X_i' = \theta(X^{\mathbf{c}^i})$ where θ is a $\mathbb{Q}(q^{1/2})$-linear automorphism of $\mathscr{F}$, and $(\mathbf{c}^1, \ldots, \mathbf{c}^r)$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^r$. Therefore

$$\mu_k(\mathscr{S}) := ((\mu_k(X_1), \ldots, \mu_k(X_r)), \ \mu_k(L), \ \mu_k(\widetilde{B})) \tag{8.11}$$

is a new quantum seed in $\mathscr{F}$, called the *mutation of $\mathscr{S}$ in direction k*. Moreover, the mutation operation is involutive, that is, $\mu_k(\mu_k(\mathscr{S})) = \mathscr{S}$.

Definition 8.1 *The quantum cluster algebra $\mathscr{A}_{q^{1/2}}(\mathscr{S})$ is the $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of the skew field $\mathscr{F}$ generated by the union of clusters of all quantum seeds obtained from $\mathscr{S}$ by any sequence of mutations.*

The following basic result is called the *quantum Laurent phenomenon*.

Theorem 8.2 ([BZ2]) *The quantum cluster algebra $\mathscr{A}_{q^{1/2}}(\mathscr{S})$ is contained in the based quantum torus generated by the quantum cluster variables of any given quantum seed $\mathscr{S}'$ mutation equivalent to $\mathscr{S}$.*

In the next sections we are going to construct a class of quantum cluster algebras attached to some categories of representations of preprojective algebras.

9 The category $\mathscr{C}_w$

We recall the main facts about the category $\mathscr{C}_w$ and its maximal rigid objects, following [**BIRS**, **GLS4**, **GLS6**].

9.1 The preprojective algebra

Let Q be a finite connected quiver without oriented cycles, with vertex set I, and arrow set Q_1. We can associate with Q a symmetric generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$, where $a_{ij} = 2$ if $i = j$, and otherwise a_{ij} is minus the number of edges between i and j in the underlying unoriented graph. We will assume that A is the Cartan matrix attached to the Kac-Moody algebra $\mathfrak{g}$, that is,

$$a_{ij} = \langle h_i, \alpha_j \rangle = (\alpha_i, \alpha_j), \qquad (i, j \in I). \tag{9.1}$$

Let $\mathbb{C}\overline{Q}$ be the path algebra of the *double quiver* $\overline{Q}$ of Q, which is obtained from Q by adding to each arrow $a: i \to j$ in Q_1 an arrow $a^*: j \to i$ pointing in the opposite direction. Let (c) be the two-sided ideal of $\mathbb{C}\overline{Q}$ generated by the element

$$c = \sum_{a \in Q_1} (a^* a - a a^*).$$

The algebra

$$\Lambda := \mathbb{C}\overline{Q}/(c)$$

is called the *preprojective algebra* of Q. Recall that for all $X,Y \in \mathrm{mod}(\Lambda)$ we have

$$\dim \mathrm{Ext}_\Lambda^1(X,Y) = \dim \mathrm{Ext}_\Lambda^1(Y,X). \tag{9.2}$$

This follows for example from the following important formula

$$\dim \mathrm{Ext}_\Lambda^1(X,Y) = \dim \mathrm{Hom}_\Lambda(X,Y) + \dim \mathrm{Hom}_\Lambda(Y,X) - (\underline{\dim}X,\ \underline{\dim}Y), \tag{9.3}$$

where we identify the dimension vector $\underline{\dim}X$ with an element of Q_+ in the standard way.

We denote by $\widehat{I}_i$ $(i \in I)$ the indecomposable injective Λ-module with socle S_i. Here, S_i is the one-dimensional simple Λ-module supported on the vertex i of Q. Note that the modules $\widehat{I}_i$ are infinite-dimensional if Q is not a Dynkin quiver.

For a Λ-module M, let $\mathrm{soc}_{(j)}(M)$ be the sum of all submodules U of M with $U \cong S_j$. For $(j_1,\dots,j_s) \in I^s$, there is a unique sequence

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s \subseteq M$$

of submodules of M such that $M_p/M_{p-1} = \mathrm{soc}_{(j_p)}(M/M_{p-1})$. Define $\mathrm{soc}_{(j_1,\dots,j_s)}(M) := M_s$. (In this definition, we do not assume that M is finite-dimensional.)

9.2 The subcategory $\mathscr{C}_w$

Let $\mathbf{i} = (i_r,\dots,i_1)$ be a reduced expression of $w \in W$. For $1 \le k \le r$, let

$$V_k := V_{\mathbf{i},k} := \mathrm{soc}_{(i_k,\dots,i_1)}\left(\widehat{I}_{i_k}\right), \tag{9.4}$$

and set $V_{\mathbf{i}} := V_1 \oplus \cdots \oplus V_r$. The module $V_{\mathbf{i}}$ is dual to the cluster-tilting object constructed in [**BIRS**, Section II.2]. Define

$$\mathscr{C}_{\mathbf{i}} := \mathrm{Fac}(V_{\mathbf{i}}) \subseteq \mathrm{mod}(\Lambda).$$

This is the full subcategory of $\mathrm{mod}(\Lambda)$ whose objects are quotient modules of a direct sum of a finite number of copies of $V_{\mathbf{i}}$. For $j \in I$, let $k_j := \max\{1 \le k \le r \mid i_k = j\}$. Define $I_{\mathbf{i},j} := V_{\mathbf{i},k_j}$ and set

$$I_{\mathbf{i}} := I_{\mathbf{i},1} \oplus \cdots \oplus I_{\mathbf{i},n}.$$

The category $\mathscr{C}_{\mathbf{i}}$ and the module $I_{\mathbf{i}}$ depend only on w, and not on the chosen reduced expression $\mathbf{i}$ of w. Therefore, we define

$$\mathscr{C}_w := \mathscr{C}_{\mathbf{i}}, \qquad I_w := I_{\mathbf{i}}.$$

Theorem 9.1 ([BIRS, Theorem II.2.8]) *For any $w \in W$, the following hold:*

(a) *$\mathscr{C}_w$ is a Frobenius category. Its stable category $\underline{\mathscr{C}}_w$ is a 2-Calabi-Yau category.*

(b) *The indecomposable $\mathscr{C}_w$-projective-injective modules are the indecomposable direct summands of I_w.*

(c) *$\mathscr{C}_w = \mathrm{Fac}(I_w)$.*

Note that in the case when Q is a Dynkin quiver, that is, $\mathfrak{g}$ is a simple Lie algebra of type A,D,E, and $w = w_0$ is the longest element in W, then $\mathscr{C}_{w_0} = \mathrm{mod}(\Lambda)$.

9.3 Maximal rigid objects

For a Λ-module T, we denote by $\text{add}(T)$ the additive closure of T, that is, the full subcategory of $\text{mod}(\Lambda)$ whose objects are isomorphic to direct sums of direct summands of T. A Λ-module T is called *rigid* if $\text{Ext}^1_\Lambda(T,T) = 0$. Let $T \in \mathscr{C}_w$ be rigid. We say that

- T is $\mathscr{C}_w$-*maximal rigid* if $\text{Ext}^1_\Lambda(T \oplus X, X) = 0$ with $X \in \mathscr{C}_w$ implies $X \in \text{add}(T)$;

- T is a $\mathscr{C}_w$-*cluster-tilting module* if $\text{Ext}^1_\Lambda(T, X) = 0$ with $X \in \mathscr{C}_w$ implies $X \in \text{add}(T)$.

Theorem 9.2 ([BIRS, Theorem I.1.8]) *For a rigid Λ-module T in $\mathscr{C}_w$ the following are equivalent:*

(a) T has r pairwise non-isomorphic indecomposable direct summands;

(b) T is $\mathscr{C}_w$-maximal rigid;

(c) T is a $\mathscr{C}_w$-cluster-tilting module.

Note that, given Theorem 9.1, the proof of [**GLS1**, Theorem 2.2] carries over to this more general situation (see [**GLS4**, Theorem 2.2] in the case when w is adaptable).

9.4 The quiver Γ_T and the matrix $\widetilde{B}_T$

Let $T = T_1 \oplus \cdots \oplus T_r$ be a $\mathscr{C}_w$-maximal rigid module, with each summand T_i indecomposable. Clearly each indecomposable $\mathscr{C}_w$-injective module $I_{\mathbf{i},j}$ is isomorphic to one of the T_k's, so, up to relabelling, we can assume that $T_{r-n+j} \cong I_{\mathbf{i},j}$ for $j = 1, \ldots, n$. Consider the endomorphism algebra $A_T := \text{End}_\Lambda(T)^{\text{op}}$. This is a basic algebra, with indecomposable projective modules

$$P_{T_i} := \text{Hom}_\Lambda(T, T_i) \ (1 \leq i \leq r). \tag{9.5}$$

The simple A_T-modules are the heads S_{T_i} of the projectives P_{T_i}. One defines a quiver Γ_T with vertex set $\{1, \ldots, r\}$, and d_{ij} arrows from i to j, where $d_{ij} = \dim \text{Ext}^1_{A_T}(S_{T_i}, S_{T_j})$. (This is known as the Gabriel quiver of A_T.) Most of the information contained in Γ_T can be encoded in an $r \times (r-n)$-matrix $\widetilde{B}_T = (b_{ij})_{1 \leq i \leq r, \ 1 \leq j \leq r-n}$, given by

$$b_{ij} = (\text{number of arrows } j \to i \text{ in } \Gamma_T) - (\text{number of arrows } i \to j \text{ in } \Gamma_T). \tag{9.6}$$

Note that $\widetilde{B}_T$ can be regarded as an exchange matrix, with skew-symmetric principal part.

The next theorem gives an explicit description of the quiver Γ_T (hence also of the matrix $\widetilde{B}_T$) for certain $\mathscr{C}_w$-maximal rigid modules. Following [**BFZ**], we define a quiver $\Gamma_{\mathbf{i}}$ as follows. The vertex set of $\Gamma_{\mathbf{i}}$ is equal to $\{1, \ldots, r\}$. For $1 \leq k \leq r$, let

$$k^- := \max\left(\{0\} \cup \{1 \leq s \leq k-1 \mid i_s = i_k\}\right), \tag{9.7}$$
$$k^+ := \min\left(\{k+1 \leq s \leq r \mid i_s = i_k\} \cup \{r+1\}\right). \tag{9.8}$$

For $1 \leq s, t \leq r$ such that $i_s \neq i_t$, there are $|a_{i_s,i_t}|$ arrows from s to t provided $t^+ \geq s^+ > t > s$. These are called the *ordinary arrows* of $\Gamma_{\mathbf{i}}$. Furthermore, for each $1 \leq s \leq r$ there is an arrow $s \to s^-$ provided $s^- > 0$. These are the *horizontal arrows* of $\Gamma_{\mathbf{i}}$.

The following result generalizes [**GLS3**, Theorem 1] (see [**GLS4**, Theorem 2.3] in the case when w is adaptable).

Theorem 9.3 ([BIRS, Theorem II.4.1]) *The Λ-module $V_{\mathbf{i}}$ is a $\mathscr{C}_w$-maximal rigid module, and we have $\Gamma_{V_{\mathbf{i}}} = \Gamma_{\mathbf{i}}$.*

9.5 Mutations of maximal rigid objects

We consider again an arbitrary $\mathscr{C}_w$-maximal rigid module T, and we use the notation of 9.4.

Theorem 9.4 ([BIRS, Theorem I.1.10]) *Let T_k be a non-projective indecomposable direct summand of T.*

(a) *There exists a unique indecomposable module $T_k^* \not\cong T_k$ such that $(T/T_k) \oplus T_k^*$ is $\mathscr{C}_w$-maximal rigid. We call $(T/T_k) \oplus T_k^*$ the* mutation of T in direction k, *and denote it by $\mu_k(T)$.*

(b) *We have $\widetilde{B}_{\mu_k(T)} = \mu_k(\widetilde{B}_T)$.*

(c) *We have $\dim \operatorname{Ext}_\Lambda^1(T_k, T_k^*) = 1$. Let $0 \to T_k \to T_k' \to T_k^* \to 0$ and $0 \to T_k^* \to T_k'' \to T_k \to 0$ be non-split short exact sequences. Then*

$$T_k' \cong \bigoplus_{b_{jk}<0} T_j^{-b_{jk}}, \qquad T_k'' \cong \bigoplus_{b_{jk}>0} T_j^{b_{jk}}.$$

Note that again, given Theorem 9.1, the proof of [**GLS1**, §2.6] carries over to this more general situation.

9.6 The modules $M[l,k]$

In this section we assume that the reduced word **i** is fixed. So for simplicity we often omit it from the notation. Thus, for $k = 1, \ldots, r$, we may write V_k instead of $V_{\mathbf{i},k}$. Moreover, we use the convention that $V_0 = 0$.

For $1 \leq k \leq l \leq r$ such that $i_k = i_l = i$, we have a natural embedding of Λ-modules $V_{k^-} \to V_l$. Following [**GLS6**, §9.8], we define $M[l,k]$ as the cokernel of this embedding, that is,

$$M[l,k] := V_l/V_{k^-}. \tag{9.9}$$

In particular, we set $M_k := M[k,k]$, and

$$M = M_{\mathbf{i}} := M_1 \oplus \cdots \oplus M_r. \tag{9.10}$$

We will use the convention that $M[l,k] = 0$ if $k > l$. Every module $M[l,k]$ is indecomposable and rigid. But note that M is *not* rigid. Define

$$k_{\min} := \min\{1 \leq s \leq r \mid i_s = i_k\}, \tag{9.11}$$
$$k_{\max} := \max\{1 \leq s \leq r \mid i_s = i_k\}. \tag{9.12}$$

Then $V_k = M[k,k_{\min}]$ corresponds to an *initial interval*. The direct sum of all modules $M[k_{\max},k]$ corresponding to *final intervals* is also a $\mathscr{C}_w$-maximal rigid module, denoted by $T_{\mathbf{i}}$.

By [**GLS6**, §10], for every module $X \in \mathscr{C}_w$, there exists a chain

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X \tag{9.13}$$

of submodules of X with $X_k/X_{k-1} \cong M_k^{m_k}$, for some uniquely determined non-negative integers m_k. The r-tuple $\mathbf{m}(X) := (m_1, \ldots, m_r)$ will be called the *M-dimension vector* of X.

9.7 Dimensions of Hom-spaces

Let $R = R_{\mathbf{i}} = (r_{kl})$ the $r \times r$-matrix with entries

$$
r_{kl} = \begin{cases} 0 & \text{if } k < l, \\ 1 & \text{if } k = l, \\ (\beta_k, \beta_l) & \text{if } k > l, \end{cases} \tag{9.14}
$$

Proposition 9.5 *Suppose that $X, Y \in \mathscr{C}_w$ satisfy $\operatorname{Ext}^1_\Lambda(X, Y) = 0$. Then*

$$
\dim \operatorname{Hom}_\Lambda(X, Y) = \mathbf{m}(X) \, R \, \mathbf{m}(Y)^T.
$$

Proof — By [**GLS6**, Proposition 10.5], for all $Y \in \mathscr{C}_w$ with M-dimension vector $\mathbf{m}(Y)$, we have

$$
\dim \operatorname{Hom}_\Lambda(V_k, Y) = \dim \operatorname{Hom}_\Lambda \left(V_k, \bigoplus_k M_k^{(\mathbf{m}(Y))_k} \right). \tag{9.15}
$$

Moreover, the M-dimension vector of V_k is given by

$$
(\mathbf{m}(V_k))_j = \begin{cases} 1 & \text{if } i_j = i_k \text{ and } j \leq k, \\ 0 & \text{else.} \end{cases} \tag{9.16}
$$

Therefore, we can restate [**GLS6**, Lemma 9.8] as

$$
\dim \operatorname{Hom}_\Lambda(V_k, Y) = \mathbf{m}(V_k) \, R \, \mathbf{m}(Y)^T. \tag{9.17}
$$

Now, for $X \in \mathscr{C}_w$ we can find a short exact sequence $0 \to V'' \to V' \to X \to 0$ with $V', V'' \in \operatorname{add}(V_{\mathbf{i}})$. Since $\operatorname{Hom}_\Lambda(V_{\mathbf{i}}, -)$ is exact on this sequence we conclude that

$$
\mathbf{m}(X) = \mathbf{m}(V') - \mathbf{m}(V''). \tag{9.18}
$$

Finally, since $\operatorname{Ext}^1_\Lambda(X, Y) = 0$, the sequence

$$
0 \to \operatorname{Hom}_\Lambda(X, Y) \to \operatorname{Hom}_\Lambda(V', Y) \to \operatorname{Hom}_\Lambda(V'', Y) \to 0
$$

is exact. Thus we can calculate

$$
\begin{aligned}
\dim \operatorname{Hom}_\Lambda(X, Y) &= \dim \operatorname{Hom}_\Lambda(V', Y) - \dim \operatorname{Hom}_\Lambda(V'', Y) \\
&= (\mathbf{m}(V') - \mathbf{m}(V'')) \, R \, \mathbf{m}(Y)^T \\
&= \mathbf{m}(X) \, R \, \mathbf{m}(Y)^T.
\end{aligned}
$$

$\square$

Lemma 9.6 *Let $1 \leq b < d \leq r$ be such that $i_b = i_d = i$. There holds*

$$
\dim \operatorname{Hom}_\Lambda(M[d, b^+], M[d^-, b]) = \mathbf{m}(M[d, b]) \, R \, \mathbf{m}(M[d^-, b])^T.
$$

Proof — We have a short exact sequence

$$0 \to M[d^-, b] \to M[d, b] \oplus M[d^-, b^+] \to M[d, b^+] \to 0$$

with

$$\mathbf{m}(M[d^-, b]) - (\mathbf{m}(M[d, b]) + \mathbf{m}(M[d^-, b^+])) + \mathbf{m}(M[d, b^+]) = 0. \tag{9.19}$$

Since

$$\mathrm{Ext}^1_\Lambda(M[d, b^+], M[d^-, b]) = 1, \quad \mathrm{Ext}^1_\Lambda(M[d, b^+], M[d, b]) = \mathrm{Ext}^1_\Lambda(M[d, b^+], M[d^-, b^+]) = 0,$$

this yields an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(M[d, b^+], M[d^-, b]) \to \mathrm{Hom}_\Lambda(M[d, b^+], M[d, b] \oplus M[d^-, b^+]) \to$$
$$\to \mathrm{Hom}_\Lambda(M[d, b^+], M[d, b^+]) \to \mathbb{C} \to 0.$$

Applying Proposition 9.5 to the second and third non-trivial terms yields the required equality, if we take into account (9.19) and the fact that $r_{bb} = 1$. $\qquad\square$

Proposition 9.7 *Let $1 \le b < d \le r$ be such that $i_b = i_d = i$. Let $j, k \in I$ with $j \neq i$ and $k \neq i$. There holds*

$$\dim \mathrm{Hom}_\Lambda(M[d, b^+], M[d^-, b]) = (\underline{\dim} V_d, \underline{\dim} M[d^-, b]) - (\underline{\dim} M[d^-, b])_i,$$

$$\dim \mathrm{Hom}_\Lambda(M[d, b], M[d^-, b^+]) = (\underline{\dim} V_d, \underline{\dim} M[d^-, b^+]) - (\underline{\dim} M[d^-, b^+])_i,$$

$$\dim \mathrm{Hom}_\Lambda(M[d^-(j), (b^-(j))^+], M[d^-(k), (b^-(k))^+]) = (\underline{\dim} V_{d^-(j)}, \underline{\dim} M[d^-(k), (b^-(k))^+])$$
$$- (\underline{\dim} M[d^-(k), (b^-(k))^+])_j.$$

Proof — By Lemma 9.6,

$$\dim \mathrm{Hom}_\Lambda(M[d, b^+], M[d^-, b]) = \dim \mathrm{Hom}_\Lambda(M[d, b], M[d^-, b]).$$

Moreover, Proposition 9.5 shows that

$$\dim \mathrm{Hom}_\Lambda(M[d, b], M[d^-, b]) = \dim \mathrm{Hom}_\Lambda(V_d, M[d^-, b]).$$

Now, it follows from (9.3) that

$$\dim \mathrm{Hom}(V_d, M[d^-, b]) = (\underline{\dim} V_d, \underline{\dim} M[d^-, b]) - (\underline{\dim} M[d^-, b])_i.$$

Indeed, V_d and $M[d^-, b]$ belong to the subcategory $\mathscr{C}_{s_{i_d} \cdots s_{i_1}}$, and V_d is projective-injective in this category, with socle S_i. Therefore

$$\dim \mathrm{Hom}(M[d^-, b], V_d) = (\underline{\dim} M[d^-, b])_i \quad \text{and} \quad \mathrm{Ext}^1_\Lambda(M[d^-, b], V_d) = 0.$$

This proves the first equation. For the remaining two equations we note that again, by Proposition 9.5, we can replace in the left hand side $M[d, b]$ by V_d, and $M[d^-(j), (b^-(j))^+]$ by $V_{d^-(j)}$. The claim follows. $\qquad\square$

10 The quantum cluster algebra associated with $\mathscr{C}_w$

10.1 The cluster algebra $\mathscr{A}(\mathscr{C}_w)$

Following [**GLS4**], [**BIRS**], [**GLS6**], we can associate with $\mathscr{C}_w$ a (classical *i.e.* not quantum) cluster algebra. This is given by the initial seed

$$\Sigma_{V_{\mathbf{i}}} := ((x_1, \ldots, x_r), \widetilde{B}_{V_{\mathbf{i}}}). \tag{10.1}$$

Although this seed depends on the choice of a reduced expression $\mathbf{i}$ for w, one can show that any two matrices $\widetilde{B}_{V_{\mathbf{i}}}$ and $\widetilde{B}_{V_{\mathbf{j}}}$ are connected by a sequence of mutations. Therefore this cluster algebra is independent of this choice, and we denote it by $\mathscr{A}(\mathscr{C}_w)$. Moreover, every seed of $\mathscr{A}(\mathscr{C}_w)$ is of the form

$$\Sigma_T = ((x_{T_1}, \ldots, x_{T_r}), \widetilde{B}_T),$$

for a unique $\mathscr{C}_w$-maximal rigid module $T = T_1 \oplus \cdots \oplus T_r$, and some Laurent polynomials $x_{T_1}, \ldots, x_{T_r}$ in the variables $x_1 = x_{V_{\mathbf{i},1}}, \ldots, x_r = x_{V_{\mathbf{i},r}}$. These modules T are those which can be reached from $V_{\mathbf{i}}$ using a sequence of mutations, and we called them *reachable*. (It is still an open problem whether every $\mathscr{C}_w$-maximal rigid module is reachable or not.) If $\mathbf{j}$ is another reduced expression for w, it is known that $V_{\mathbf{j}}$ is reachable from $V_{\mathbf{i}}$ [**BIRS**]. Therefore, the collection of reachable $\mathscr{C}_w$-maximal rigid modules does not depend on the choice of $\mathbf{i}$.

It was shown in [**GLS4**, **GLS6**] that there is a natural isomorphism from $\mathscr{A}(\mathscr{C}_w)$ to the coordinate ring $\mathbb{C}[N(w)]$, mapping the cluster monomials to a subset of Lusztig's dual semicanonical basis of $\mathbb{C}[N(w)]$.

10.2 The matrix L_T

Let $T = T_1 \oplus \cdots \oplus T_r$ be a $\mathscr{C}_w$-maximal rigid module as in §9.4. Let $L_T = (\lambda_{ij})$ be the $r \times r$-matrix with entries

$$\lambda_{ij} := \dim \operatorname{Hom}_\Lambda(T_i, T_j) - \dim \operatorname{Hom}_\Lambda(T_j, T_i), \qquad (1 \leq i, j \leq r). \tag{10.2}$$

Note that L_T is skew-symmetric. From now on we will use the following convenient shorthand notation. Given two Λ-modules X and Y, we will write

$$[X, Y] := \dim \operatorname{Hom}_\Lambda(X, Y), \qquad [X, Y]^1 := \dim \operatorname{Ext}_\Lambda^1(X, Y). \tag{10.3}$$

Thus, we shall write $\lambda_{ij} = [T_i, T_j] - [T_j, T_i]$.

Proposition 10.1 *The pair $(L_T, \widetilde{B}_T)$ is compatible.*

Proof — For $1 \leq j \leq r - n$, and $1 \leq i, k \leq r$, by Theorem 9.4 (c) we have:

$$\sum_{k=1}^r b_{kj} \lambda_{ki} = [T_i, T_j'] + [T_j'', T_i] - [T_j', T_i] - [T_i, T_j''].$$

Let us first assume that $i \neq j$. Applying the functor $\operatorname{Hom}_\Lambda(T_i, -)$ to the short exact sequence

$$0 \to T_j \to T_j' \to T_j^* \to 0,$$

and taking into account that $[T_i, T_j]^1 = 0$, we get a short exact sequence

$$0 \to \mathrm{Hom}_\Lambda(T_i, T_j) \to \mathrm{Hom}_\Lambda(T_i, T_j') \to \mathrm{Hom}_\Lambda(T_i, T_j^*) \to 0,$$

therefore

$$[T_i, T_j'] = [T_i, T_j^*] + [T_i, T_j]. \tag{10.4}$$

Similarly, applying the functor $\mathrm{Hom}_\Lambda(-, T_i)$ to the same short exact sequence and taking into account that $[T_j^*, T_i]^1 = 0$, we get a short exact sequence

$$0 \to \mathrm{Hom}_\Lambda(T_j^*, T_i) \to \mathrm{Hom}_\Lambda(T_j', T_i) \to \mathrm{Hom}_\Lambda(T_j, T_i) \to 0,$$

therefore

$$[T_j', T_i] = [T_j^*, T_i] + [T_j, T_i]. \tag{10.5}$$

It follows that

$$[T_i, T_j'] - [T_j', T_i] = [T_i, T_j^*] - [T_j^*, T_i] + \lambda_{ij}. \tag{10.6}$$

Arguing similarly with the short exact sequence

$$0 \to T_j^* \to T_j'' \to T_j \to 0,$$

we obtain

$$[T_j'', T_i] - [T_i, T_j''] = [T_j^*, T_i] - [T_i, T_j^*] + \lambda_{ji}. \tag{10.7}$$

Hence $\sum_{k=1}^{r} b_{kj}\lambda_{ki} = \lambda_{ij} + \lambda_{ji} = 0$.

Assume now that $i = j$. Using that $[T_j, T_j^*]^1 = 1$ and $[T_j, T_j]^1 = [T_j, T_j']^1 = [T_j, T_j'']^1 = 0$, and arguing as above, we easily obtain the relations

$$[T_j, T_j'] = [T_j, T_j^*] + [T_j, T_j], \tag{10.8}$$

$$[T_j', T_j] = [T_j^*, T_j] + [T_j, T_j] - 1, \tag{10.9}$$

$$[T_j'', T_j] = [T_j^*, T_j] + [T_j, T_j], \tag{10.10}$$

$$[T_j, T_j''] = [T_j, T_j^*] + [T_j, T_j] - 1. \tag{10.11}$$

It follows that $\sum_{k=1}^{r} b_{kj}\lambda_{kj} = 2$. Thus, in general,

$$\sum_{k=1}^{r} b_{kj}\lambda_{ki} = 2\delta_{ij},$$

which proves the proposition. $\qquad\qquad\square$

Let $k \le r - n$, so that the mutation $\mu_k(T)$ is well-defined.

Proposition 10.2 *We have $\mu_k(L_T) = L_{\mu_k(T)}$.*

Proof — Put $\mu_k(L_T) = (\lambda_{ij}')$. By definition, $\lambda_{ij}' = \lambda_{ij}$ if i or j is different from k. Similarly, since T and $\mu_k(T)$ differ only by the replacement of T_k by T_k^*, all entries in $L_{\mu_k(T)}$ not situated on row k or column k are equal to the corresponding entries of L_T.

If $i = k$, we have by definition of $\mu_k(L_T)$ and by Theorem 9.4 (c)

$$\lambda'_{kj} = \sum_{s=1}^{r} e_{sk}\lambda_{sj} = [T'_k, T_j] - [T_j, T'_k] - [T_k, T_j] + [T_j, T_k] - \lambda_{kj}.$$

Thus, by (10.6), we get

$$\lambda'_{kj} = [T^*_k, T_j] - [T_j, T^*_k],$$

as claimed. The case $j = k$ follows by skew-symmetry. □

Let $H_{\mathbf{i}} = (h_{kl})$ be the $r \times r$-matrix with entries

$$h_{kl} = \begin{cases} 1 & \text{if } l = k^{(-m)} \text{ for some } m \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{10.12}$$

Proposition 10.3 *The matrix $L_{V_{\mathbf{i}}}$ has the following explicit expression*

$$L_{V_{\mathbf{i}}} = H_{\mathbf{i}}(R_{\mathbf{i}} - R_{\mathbf{i}}^T)H_{\mathbf{i}}^T.$$

Proof— This follows immediately from Proposition 9.5, if we note that the kth row of $H_{\mathbf{i}}$ is equal to the M-dimension vector $\mathbf{m}(V_{\mathbf{i},k})$. □

10.3   The quantum cluster algebra $\mathscr{A}_{q^{1/2}}(\mathscr{C}_w)$

Proposition 10.1 and Proposition 10.2 show that the family $(L_T, \widetilde{B}_T)$, where T ranges over all $\mathscr{C}_w$-maximal rigid modules reachable from $V_{\mathbf{i}}$, gives rise to a quantum analogue of the cluster algebra $\mathscr{A}(\mathscr{C}_w)$. We shall denote it by $\mathscr{A}_{q^{1/2}}(\mathscr{C}_w)$. For every reduced expression $\mathbf{i}$ of w, an explicit initial quantum seed is given by

$$\mathscr{S}_{V_{\mathbf{i}}} := ((X_{V_{\mathbf{i},1}}, \ldots, X_{V_{\mathbf{i},r}}), L_{V_{\mathbf{i}}}, \widetilde{B}_{V_{\mathbf{i}}}),$$

where the matrices $L_{V_{\mathbf{i}}}$ and $\widetilde{B}_{V_{\mathbf{i}}}$ have been computed in Proposition 10.3, and in Section 9.4.

The quantum seed corresponding to a reachable $\mathscr{C}_w$-maximal rigid module $T = T_1 \oplus \cdots \oplus T_r$ will be denoted by

$$\mathscr{S}_T = ((X_{T_1}, \ldots, X_{T_r}), L_T, \widetilde{B}_T). \tag{10.13}$$

For every $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r$, we have a rigid module $T^{\mathbf{a}} := T_1^{a_1} \oplus \cdots \oplus T_r^{a_r}$ in the additive closure add (T) of T. Following (8.2) and writing $L_T = (\lambda_{ij})$, we put

$$X_{T^{\mathbf{a}}} = X^{\mathbf{a}} := q^{\frac{1}{2}\sum_{i>j} a_i a_j \lambda_{ij}} X_{T_1}^{a_1} \cdots X_{T_r}^{a_r}. \tag{10.14}$$

Thus, denoting by $\mathscr{R}_w$ the set of rigid modules R in the additive closure of some reachable $\mathscr{C}_w$-maximal rigid module, we obtain a canonical labelling

$$X_R, \qquad (R \in \mathscr{R}_w). \tag{10.15}$$

of the quantum cluster monomials of $\mathscr{A}_{q^{1/2}}(\mathscr{C}_w)$.

10.4 The elements Y_R

It will be convenient in our setting to proceed to a slight rescaling of the elements X_R. For $R \in \mathscr{R}_w$, we define

$$Y_R := q^{-[R,R]/2} X_R. \tag{10.16}$$

In particular, writing $R = T^{\mathbf{a}}$ as above, an easy calculation gives

$$Y_R = q^{-\alpha(R)} Y_{T_1}^{a_1} \cdots Y_{T_r}^{a_r}, \tag{10.17}$$

where

$$\alpha(R) := \sum_{i<j} a_i a_j [T_i, T_j] + \sum_i \binom{a_i}{2} [T_i, T_i]. \tag{10.18}$$

Note that $\alpha(R)$ is an integer (not a half-integer), so that Y_R belongs to $\mathbb{Z}[q^{\pm 1}][Y_{T_1}, \cdots, Y_{T_r}]$. Moreover, we have the following easy lemma.

Lemma 10.4 *Let T be a reachable $\mathscr{C}_w$-maximal rigid module. For any R, S in $\mathrm{add}\,(T)$ we have*

$$Y_R Y_S = q^{[R,S]} Y_{R \oplus S}.$$

Proof — Write $R = T_1^{a_1} \oplus \cdots \oplus T_r^{a_r}$ and $S = T_1^{b_1} \oplus \cdots \oplus T_r^{b_r}$. We have

$$Y_R Y_S = q^{-\alpha(R)-\alpha(S)} Y_{T_1}^{a_1} \cdots Y_{T_r}^{a_r} Y_{T_1}^{b_1} \cdots Y_{T_r}^{b_r} = q^{-\alpha(R)-\alpha(S)+\sum_{i>j} a_i b_j \lambda_{ij}} Y_{T_1}^{a_1+b_1} \cdots Y_{T_r}^{a_r+b_r}.$$

On the other hand, using the obvious identity

$$\binom{a+b}{2} = \binom{a}{2} + \binom{b}{2} + ab,$$

we see that

$$\alpha(R \oplus S) = \alpha(R) + \alpha(S) + \sum_{i<j} (a_i b_j + a_j b_i)[T_i, T_j] + \sum_i a_i b_i [T_i, T_i].$$

Hence, taking into account (10.2),

$$
\begin{aligned}
-\alpha(R) - \alpha(S) + \sum_{i>j} a_i b_j \lambda_{ij} &= -\alpha(R \oplus S) + \sum_i a_i b_i [T_i, T_i] + \sum_{i>j} a_i b_j [T_i, T_j] + \sum_{i<j} a_i b_j [T_i, T_j] \\
&= -\alpha(R \oplus S) + [R, S],
\end{aligned}
$$

and the result follows. $\qquad\square$

Note that because of (9.3) and the fact that every $R \in \mathrm{add}\,(T)$ is rigid, we have

$$[R, R] = (\underline{\dim} R, \ \underline{\dim} R)/2, \tag{10.19}$$

hence the exponent of q in (10.16) depends only on the dimension vector $\underline{\dim} R$ of R.

10.5 Quantum mutations

We now rewrite formulas (8.8) (8.9) for quantum mutations in $\mathscr{A}_{q^{1/2}}(\mathscr{C}_w)$, using the rescaled quantum cluster monomials Y_R.

Let T be a $\mathscr{C}_w$-maximal rigid module, and let T_k be a non-projective indecomposable direct summand of T. Let $\mu_k(T) = (T/T_k) \oplus T_k^*$ be the mutation of T in direction k. By Theorem 9.4, we have two short exact sequences

$$0 \to T_k \to T_k' \to T_k^* \to 0, \qquad 0 \to T_k^* \to T_k'' \to T_k \to 0, \tag{10.20}$$

with $T_k', T_k'' \in \mathrm{add}\,(T/T_k)$. The quantum exchange relation between the quantum cluster variables Y_{T_k} and $Y_{T_k^*}$ can be written as follows:

Proposition 10.5 *With the above notation, we have*

$$Y_{T_k^*} Y_{T_k} = q^{[T_k^*,T_k]}(q^{-1} Y_{T_k'} + Y_{T_k''}).$$

Proof — We have

$$
\begin{aligned}
Y_{T_k^*} Y_{T_k} &= q^{-([T_k^*,T_k^*]+[T_k,T_k])/2} X_{T_k^*} X_{T_k} \\
&= q^{-([T_k^*,T_k^*]+[T_k,T_k])/2} \left(X^{\mathbf{a}'} + X^{\mathbf{a}''} \right) X_{T_k},
\end{aligned}
$$

where, if we write

$$T_k' = \bigoplus_{i \neq k} T_i^{a_i'}, \qquad T_k'' = \bigoplus_{i \neq k} T_i^{a_i''},$$

the multi-exponents $\mathbf{a}'$ and $\mathbf{a}''$ are given by

$$\mathbf{a}' = (a_1' \dots, a_{k-1}', -1, a_{k+1}', \dots, a_r'), \quad \mathbf{a}'' = (a_1'' \dots, a_{k-1}'', -1, a_{k+1}'', \dots, a_r'').$$

Using (8.1), (8.2), and (10.2), one obtains easily that

$$X^{\mathbf{a}'} X_{T_k} = q^{([T_k',T_k]-[T_k,T_k'])/2} X_{T_k'}, \qquad X^{\mathbf{a}''} X_{T_k} = q^{([T_k'',T_k]-[T_k,T_k''])/2} X_{T_k''}.$$

Now, using twice (9.3), we have

$$
\begin{aligned}
[T_k', T_k'] &= (\underline{\dim}\,T_k', \underline{\dim}\,T_k')/2 & (10.21) \\
&= (\underline{\dim}\,T_k + \underline{\dim}\,T_k^*, \underline{\dim}\,T_k + \underline{\dim}\,T_k^*)/2 & (10.22) \\
&= [T_k, T_k] + [T_k^*, T_k^*] + (\underline{\dim}\,T_k, \underline{\dim}\,T_k^*) & (10.23) \\
&= [T_k, T_k] + [T_k^*, T_k^*] + [T_k, T_k^*] + [T_k^*, T_k] - 1, & (10.24)
\end{aligned}
$$

and this is also equal to $[T_k'', T_k'']$. Therefore, the exponent of $q^{1/2}$ in front of $Y_{T_k'}$ in the product $Y_{T_k^*} Y_{T_k}$ is equal to

$$
\begin{aligned}
[T_k', T_k] - [T_k, T_k'] + [T_k', T_k'] - [T_k^*, T_k^*] - [T_k, T_k] &= [T_k, T_k^*] + [T_k^*, T_k] - 1 - [T_k, T_k'] + [T_k', T_k] \\
&= 2([T_k^*, T_k] - 1),
\end{aligned}
$$

where the second equality follows from (10.8) and (10.9). Similarly, using (10.10) and (10.11), we see that the exponent of $q^{1/2}$ in front of $Y_{T_k''}$ in the product $Y_{T_k^*} Y_{T_k}$ is equal to $2[T_k^*, T_k]$, as claimed. $\square$

10.6 The rescaled quantum cluster algebras $\mathscr{A}_q(\mathscr{C}_w)$ and $\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)$

By definition, $\mathscr{A}_{q^{1/2}}(\mathscr{C}_w)$ is a $\mathbb{Z}[q^{\pm 1/2}]$-algebra. It follows from Lemma 10.4 and Proposition 10.5 that if we replace the quantum cluster variables X_{T_i} by their rescaled versions Y_{T_i} we no longer need half-integral powers of q. So we are led to introduce the rescaled quantum cluster algebra $\mathscr{A}_q(\mathscr{C}_w)$. This is defined as the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $\mathscr{A}_{q^{1/2}}(\mathscr{C}_w)$ generated by the elements Y_{T_i}, where T_i ranges over the indecomposable direct summands of all reachable $\mathscr{C}_w$-maximal rigid modules.

It turns out that in the sequel, in order to have a coefficient ring which is a principal ideal domain, it will be convenient to slightly extend coefficients from $\mathbb{Z}[q^{\pm 1}]$ to $\mathbb{A} = \mathbb{Q}[q^{\pm 1}]$. We will denote by

$$\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w) := \mathbb{A} \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathscr{A}_q(\mathscr{C}_w)$$

the corresponding quantum cluster algebra. Of course, the based quantum tori of $\mathscr{A}_q(\mathscr{C}_w)$ (*resp.* $\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)$) will be defined over $\mathbb{Z}[q^{\pm 1}]$ (*resp.* over $\mathbb{A}$).

10.7 The involution σ

We now define an involution of $\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)$, which will turn out to be related to the involution σ of $U_q(\mathfrak{n})$ defined in §6.2. For this reason we shall also denote it by σ. This involution is a twisted-bar involution, as defined by Berenstein and Zelevinsky [**BZ2**, §6].

We first define an additive group automorphism σ of the $\mathbb{A}$-module $\mathbb{A}[Y_R \mid R \in \mathrm{add}\,(V_{\mathbf{i}})]$ by setting

$$\sigma(f(q)Y_R) \quad = \quad f(q^{-1})q^{[R,R]-\dim R}Y_R, \qquad (f \in \mathbb{A},\ R \in \mathrm{add}\,(V_{\mathbf{i}})). \tag{10.25}$$

Clearly, σ is an involution.

Lemma 10.6 σ *is a ring anti-automorphism.*

Proof — We have

$$
\begin{aligned}
\sigma(Y_R Y_S) &= q^{-[R,S]}\sigma(Y_{R\oplus S}) \\
&= q^{-[R,S]-\dim(R\oplus S)+[R\oplus S,\ R\oplus S]}Y_{R\oplus S} \\
&= q^{-\dim R-\dim S+[R,R]+[S,S]+[S,R]}Y_{R\oplus S} \\
&= q^{-\dim R-\dim S+[R,R]+[S,S]}Y_S Y_R \\
&= \sigma(Y_S)\sigma(Y_R).
\end{aligned}
$$

$\square$

The involution σ extends to an anti-automorphism of the based quantum torus

$$\mathscr{V}_{\mathbf{i}} := \mathbb{A}[Y_{V_i}^{\pm 1} \mid 1 \le i \le r] \tag{10.26}$$

which we still denote by σ. Moreover, we have

Lemma 10.7 *For every indecomposable reachable rigid module U we have*

$$\sigma(Y_U) = q^{[U,U]-\dim U}Y_U.$$

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Proof — By induction on the length of the mutation sequence, we may assume that the result holds for all indecomposable direct summands T_j of a $\mathscr{C}_w$-maximal rigid module T. We then have to check that it also holds for $U = T_k^* = \mu_k(T_k)$. By Proposition 10.5, we have

$$q^{[T_k,T_k]-\dim T_k} Y_{T_k} \sigma(Y_{T_k^*}) = q^{-[T_k^*,T_k]} \left(q^{[T_k',T_k']-\dim T_k'+1} Y_{T_k'} + q^{[T_k'',T_k'']-\dim T_k''} Y_{T_k''} \right).$$

Using that $\dim T_k' = \dim T_k'' = \dim T_k + \dim T_k^*$, and (10.24), this becomes

$$Y_{T_k} \sigma(Y_{T_k^*}) = q^{[T_k^*,T_k^*]-\dim T_k^*+[T_k,T_k^*]}(Y_{T_k'} + q^{-1}Y_{T_k''}) = q^{[T_k^*,T_k^*]-\dim T_k^*} Y_{T_k} Y_{T_k^*},$$

where the last equality comes again from Proposition 10.5, since quantum mutations are involutive. It follows that $\sigma(Y_{T_k^*}) = q^{[T_k^*,T_k^*]-\dim T_k^*} Y_{T_k^*}$, as claimed, and this proves the lemma. $\qquad\square$

By Lemma 10.7, σ restricts to an automorphism of $\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)$, that we again denote by σ. Moreover, for any quantum cluster monomial, that is for every element Y_U where U is a reachable, non necessarily indecomposable, rigid module in $\mathscr{C}_w$, there holds

$$\sigma(Y_U) = q^{[U,U]-\dim U} Y_U = q^{(\underline{\dim}\,U,\,\underline{\dim}\,U)/2-\dim U} Y_U = q^{N(\underline{\dim}\,U)} Y_U, \qquad (10.27)$$

where, for $\beta \in Q_+$, $N(\beta)$ is defined in (6.6).

11 Based quantum tori

In this section, we fix again a reduced word $\mathbf{i} := (i_r, \ldots, i_1)$ for w, and we set

$$\lambda_k = s_{i_1} \cdots s_{i_k}(\varpi_{i_k}), \qquad (k \le r). \qquad (11.1)$$

11.1 Commutation relations

The next lemma is a particular case of [**BZ2**, Theorem 10.1]. We include a brief proof for the convenience of the reader.

Lemma 11.1 *For $1 \le k < l \le r$, we have in $A_q(\mathfrak{g})$:*

$$\Delta_{\varpi_{i_k},\lambda_k}\Delta_{\varpi_{i_l},\lambda_l} = q^{(\varpi_{i_k},\varpi_{i_l})-(\lambda_k,\lambda_l)}\Delta_{\varpi_{i_l},\lambda_l}\Delta_{\varpi_{i_k},\lambda_k}.$$

Proof — Since $k < l$, we have $\lambda_l = s_{i_1} \cdots s_{i_k}(v)$, where $v = s_{i_{k+1}} \cdots s_{i_l}(\varpi_{i_l})$. For $x \in U_q(\mathfrak{n})$ and $y \in U_q(\mathfrak{n}_-)$, we have

$$x \cdot \Delta_{\varpi_{i_k},\varpi_{i_k}} = \varepsilon(x)\Delta_{\varpi_{i_k},\varpi_{i_k}}, \qquad \Delta_{\varpi_{i_l},v} \cdot y = \varepsilon(y)\Delta_{\varpi_{i_l},v}.$$

Therefore, using again [**BZ2**, Lemma 10.2] as in the proof of Lemma 3.6, we obtain that

$$\Delta_{\varpi_{i_k},\varpi_{i_k}}\Delta_{\varpi_{i_l},v} = q^{(\varpi_{i_k},\varpi_{i_l})-(\varpi_{i_k},v)}\Delta_{\varpi_{i_l},v}\Delta_{\varpi_{i_k},\varpi_{i_k}}.$$

Now, using $l - k$ times Lemma 3.4, we deduce from this equality that

$$\Delta_{\varpi_{i_k},s_{i_1}\cdots s_{i_k}(\varpi_{i_k})}\Delta_{\varpi_{i_l},s_{i_1}\cdots s_{i_k}(v)} = q^{(\varpi_{i_k},\varpi_{i_l})-(\varpi_{i_k},v)}\Delta_{\varpi_{i_l},s_{i_1}\cdots s_{i_k}(v)}\Delta_{\varpi_{i_k},s_{i_1}\cdots s_{i_k}(\varpi_{i_k})},$$

and taking into account that $(\varpi_{i_k}, v) = (s_{i_1} \cdots s_{i_k}(\varpi_{i_k}), s_{i_1} \cdots s_{i_k}(v)) = (\lambda_k, \lambda_l)$, we get the claimed equality. $\qquad\square$

Lemma 11.2 *For $1 \leq k < l \leq r$, we have in $A_q(\mathfrak{n})$:*

$$D_{\varpi_{i_k}, \lambda_k} D_{\varpi_{i_l}, \lambda_l} = q^{(\varpi_{i_k} - \lambda_k, \ \varpi_{i_l} + \lambda_l)} D_{\varpi_{i_l}, \lambda_l} D_{\varpi_{i_k}, \lambda_k}.$$

Proof — This follows from Lemma 11.1 using the same type of calculations as in the proof of Lemma 5.2. We leave the easy verification to the reader. $\qquad\square$

Lemma 11.3 *For $1 \leq k < l \leq r$, we have:*

$$(\varpi_{i_k} - \lambda_k, \ \varpi_{i_l} + \lambda_l) = [V_{\mathbf{i},k}, V_{\mathbf{i},l}] - [V_{\mathbf{i},l}, V_{\mathbf{i},k}].$$

Proof — We have

$$(\varpi_{i_k} - \lambda_k, \ \varpi_{i_l} + \lambda_l) = (\underline{\dim}\, V_{\mathbf{i},k}, \ 2\varpi_{i_l} - \underline{\dim}\, V_{\mathbf{i},l}).$$

Since $[V_{\mathbf{i},k}, V_{\mathbf{i},l}]^1 = 0$, using (9.3) we have

$$(\underline{\dim}\, V_{\mathbf{i},k}, \ \underline{\dim}\, V_{\mathbf{i},l}) = [V_{\mathbf{i},k}, V_{\mathbf{i},l}] + [V_{\mathbf{i},l}, V_{\mathbf{i},k}].$$

Hence,

$$(\varpi_{i_k} - \lambda_k, \ \varpi_{i_l} + \lambda_l) = 2(\underline{\dim}\, V_{\mathbf{i},k}, \ \varpi_{i_l}) - [V_{\mathbf{i},k}, V_{\mathbf{i},l}] - [V_{\mathbf{i},l}, V_{\mathbf{i},k}],$$

and we are reduced to show that

$$(\underline{\dim}\, V_{\mathbf{i},k}, \ \varpi_{i_l}) = [V_{\mathbf{i},k}, V_{\mathbf{i},l}], \qquad (k < l). \qquad (11.2)$$

By Proposition 10.3, the right-hand side is equal to

$$\sum_{m \leq 0} \left(\sum_{\substack{s < k^{(m)} \\ i_s = i_l}} (\beta_{k^{(m)}}, \beta_s) + \delta_{i_k i_l} \right).$$

Since the left-hand side is equal to $\sum_{m \leq 0} (\beta_{k^{(m)}}, \varpi_{i_l})$, it is enough to show that for every $m \geq 0$,

$$(\beta_{k^{(m)}}, \varpi_{i_l}) = \sum_{\substack{s < k^{(m)} \\ i_s = i_l}} (\beta_{k^{(m)}}, \beta_s) + \delta_{i_k i_l}. \qquad (11.3)$$

Let $t := \max\{s < k^{(m)} \mid i_s = i_l\}$. Eq. (11.3) can be rewritten

$$(\beta_{k^{(m)}}, \lambda_t) = \delta_{i_k i_l}. \qquad (11.4)$$

But the left-hand side of (11.4) is equal to

$$\left(s_{i_1} \cdots s_{i_{k^{(m)}-1}}(\alpha_{i_k}), \ s_{i_1} \cdots s_{i_t}(\varpi_{i_l}) \right) = \left(s_{i_1} \cdots s_{i_{k^{(m)}-1}}(\alpha_{i_k}), \ s_{i_1} \cdots s_{i_{k^{(m)}-1}}(\varpi_{i_l}) \right) = (\alpha_{i_k}, \varpi_{i_l}) = \delta_{i_k i_l},$$

so (11.4) holds, and this proves (11.2). $\qquad\square$

11.2 An isomorphism of based quantum tori

By Proposition 7.5, the quantum flag minors $D_{\varpi_{i_k}, \lambda_k} = D(0, k)$ belong to $F_q(\mathfrak{n}(w))$. Let $\mathscr{T}_{\mathbf{i}}$ be the $\mathbb{A}$-subalgebra of $F_q(\mathfrak{n}(w))$ generated by the $D_{\varpi_{i_k}, \lambda_k}^{\pm 1}$ $(1 \le k \le r)$.

Lemma 11.4 *The algebra $\mathscr{T}_{\mathbf{i}}$ is a based quantum torus over $\mathbb{A}$.*

Proof — By Lemma 11.2, the generators $D_{\varpi_{i_k}, \lambda_k}$ pairwise q-commute. So we only have to show that the monomials

$$D^{\mathbf{a}} := \overrightarrow{\prod_k} D_{\varpi_{i_k}, \lambda_k}^{a_k}, \qquad (\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{N}^r),$$

are linearly independent over $\mathbb{A}$. Suppose that we have a non-trivial relation

$$\sum_{\mathbf{a}} \gamma_{\mathbf{a}}(q) D^{\mathbf{a}} = 0,$$

for some $\gamma_{\mathbf{a}}(q) \in \mathbb{A}$. Dividing this equation (if necessary) by the largest power of $q - 1$ which divides all the coefficients $\gamma_{\mathbf{a}}(q)$, we may assume that at least one of these coefficients is not divisible by $q - 1$. Using 6.3, we see that by specializing this identity at $q = 1$ we get a non-trivial $\mathbb{C}$-linear relation between monomials in the corresponding classical flag minors of $\mathbb{C}[N]$. But all these monomials belong to the dual semicanonical basis of $\mathbb{C}[N]$ (see [**GLS6**, Corollary 13.3]), hence they are linearly independent, a contradiction. $\qquad\square$

We note that Lemma 11.4 also follows from [**Ki**, Theorem 6.20].

Proposition 11.5 *The assignment $Y_{V_{\mathbf{i},k}} \mapsto D_{\varpi_{i_k}, \lambda_k}$ $(1 \le k \le r)$ extends to an algebra isomorphism from $\mathscr{V}_{\mathbf{i}}$ to the based quantum torus $\mathscr{T}_{\mathbf{i}}$.*

Proof — By definition of the cluster algebra $\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)$, the elements $Y_{V_{\mathbf{i},k}}^{\pm 1}$ generate a based quantum torus over $\mathbb{A}$, with q-commutation relations

$$Y_{V_{\mathbf{i},k}} Y_{V_{\mathbf{i},l}} = q^{[V_{\mathbf{i},k}, V_{\mathbf{i},l}] - [V_{\mathbf{i},l}, V_{\mathbf{i},k}]} Y_{V_{\mathbf{i},l}} Y_{V_{\mathbf{i},k}}, \qquad (1 \le k < l \le r).$$

The proposition then follows immediately from Lemma 11.2 and Lemma 11.3. $\qquad\square$

12 Cluster structures on quantum coordinate rings

12.1 Cluster structures on quantum coordinate rings of unipotent subgroups

Consider the following diagram of homomorphisms of $\mathbb{A}$-algebras:

$$\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w) \longrightarrow \mathscr{V}_{\mathbf{i}} \xrightarrow{\sim} \mathscr{T}_{\mathbf{i}} \longrightarrow F_q(\mathfrak{n}(w)). \qquad (12.1)$$

Here, the first arrow denotes the natural embedding given by the (quantum) Laurent phenomenon [**BZ2**], the second arrow is the isomorphism of Proposition 11.5, and the third arrow is the natural embedding. The composition $\kappa \colon \mathscr{A}_{\mathbb{A}}(\mathscr{C}_w) \longrightarrow F_q(\mathfrak{n}(w))$ of these maps is therefore injective. Recall the notation $A_{\mathbb{A}}(\mathfrak{n}(w))$ of §7.6.

Proposition 12.1 *The image $\kappa(\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w))$ contains $A_{\mathbb{A}}(\mathfrak{n}(w))$.*

Proof — Since κ is an algebra map, it is enough to show that its image contains the dual PBW-generators $E^*(\beta_k)$ $(1 \leq k \leq r)$.

It was shown in [**GLS6**, §13.1] that there is an explicit sequence of mutations of $\mathscr{C}_w$-maximal rigid modules starting from $V_{\mathbf{i}}$ and ending in $T_{\mathbf{i}}$. Each step of this sequence consists of the mutation of a module $M[d^-,b]$ into a module $M[d,b^+]$, for some $1 \leq b < d \leq r$ with $i_b = i_d = i$. The corresponding pair of short exact sequences is

$$0 \to M[d^-,b] \to M[d^-,b^+] \oplus M[d,b] \to M[d,b^+] \to 0,$$

$$0 \to M[d,b^+] \to \bigoplus_{j \neq i} M[d^-(j),(b^-(j))^+]^{-a_{ij}} \to M[d^-,b] \to 0.$$

Write

$$T'_{bd} := M[d^-,b^+] \oplus M[d,b]$$

and

$$T''_{bd} := \bigoplus_{j \neq i} M[d^-(j),(b^-(j))^+]^{-a_{ij}}.$$

Then, by Proposition 10.5, in $\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)$ we have the corresponding mutation relation:

$$Y_{M[d,b^+]}Y_{M[d^-,b]} = q^{[M[d,b^+],M[d^-,b]]}\left(q^{-1}Y_{T'_{bd}} + Y_{T''_{bd}}\right). \tag{12.2}$$

Moreover,

$$Y_{T'_{bd}} = q^{-\alpha(T'_{bd})}Y_{M[d^-,b^+]}Y_{M[d,b]}, \qquad Y_{T''_{bd}} = q^{-\alpha(T''_{bd})}\overrightarrow{\prod_{j \neq i}}Y_{M[d^-(j),(b^-(j))^+]}^{-a_{ij}}$$

where $\alpha(T'_{bd}) = [M[d^-,b^+], M[d,b]]$, and

$$\alpha(T''_{bd}) = \sum_{j<k} a_{ij}a_{ik}[M[d^-(j),(b^-(j))^+],M[d^-(k),(b^-(k))^+]]$$

$$+ \sum_{j \neq i}\binom{-a_{ij}}{2}[M[d^-(j),(b^-(j))^+],M[d^-(j),(b^-(j))^+]].$$

Note that for $1 \leq k \leq l \leq r$, and $i_k = i_l = j$, one has

$$\underline{\dim}M[l,k] = \mu(k^-,j) - \mu(l,j). \tag{12.3}$$

Therefore, using Proposition 9.7 and the notation of Proposition 5.5, we see that

$$[M[d,b^+], M[d^-,b]] = (\underline{\dim}V_d, \underline{\dim}M[d^-,b]) - (\underline{\dim}M[d^-,b])_i$$

$$= (\varpi_i - \mu(d,i), \mu(b^-,i) - \mu(d^-,i)) - (\varpi_i, \mu(b^-,i) - \mu(d^-,i))$$

$$= -(\mu(d,i), \mu(b^-,i) - \mu(d^-,i))$$

$$= -A.$$

Similarly we obtain that

$$\alpha(T'_{bd}) = -B, \qquad \alpha(T''_{bd}) = -C.$$

Therefore (12.2) can be rewritten as

$$q^A Y_{M[d,b^+]} Y_{M[d^-,b]} = q^{-1+B} Y_{M[d^-,b^+]} Y_{M[d,b]} + q^C \overrightarrow{\prod_{j \neq i}} Y_{M[d^-(j),(b^-(j))^+]}^{-a_{ij}}. \tag{12.4}$$

We observe that this has exactly the same form as (5.14). By definition of κ one has

$$\kappa(Y_{V_{i,k}}) = \kappa(Y_{M[k,k_{\min}]}) = D(0,k).$$

Hence all the initial variables of the systems of equations (12.4) and (5.14) are matched under the algebra homomorphism κ. Therefore, by induction, it follows that $\kappa(Y_{M[d,b^+]}) = D(b,d)$ for every $b < d$. In particular, $\kappa(Y_{M[k,k]}) = D(k^-, k) = E^*(\beta_k)$, by Proposition 7.4. $\qquad \square$

Proposition 12.2 *The algebra $\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)$ is a Q_+-graded free $\mathbb{A}$-module, with homogeneous components of finite rank. Moreover,*

$$\mathrm{rk}\left(\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)_\alpha\right) = \mathrm{rk}\left(A_{\mathbb{A}}(\mathfrak{n}(w))_\alpha\right), \qquad (\alpha \in Q_+).$$

Proof — The $\mathbb{A}$-module $\mathscr{V}_\mathbf{i}$ is free, hence its submodule $\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)$ is projective, and therefore free since $\mathbb{A}$ is a principal ideal domain. It has a natural Q_+-grading given by

$$\deg Y_R := \underline{\dim} R$$

for every indecomposable reachable rigid module R in $\mathscr{C}_w$. The rank of $\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)_\alpha$ is equal to the dimension of the $\mathbb{C}$-vector space $\mathbb{C} \otimes_{\mathbb{A}} \mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)_\alpha$, which is equal to the dimension of the corresponding homogeneous component of the (classical) cluster algebra $\mathbb{C} \otimes_{\mathbb{Z}} \mathscr{A}(\mathscr{C}_w)$. Now, by [**GLS6**], this is equal to the dimension of $\mathbb{C}[N(w)]_\alpha$, that is, by §7.6, to the rank of $A_{\mathbb{A}}(\mathfrak{n}(w))_\alpha$. $\square$

Let $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w) := \mathbb{Q}(q) \otimes_{\mathbb{A}} \mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)$. The $\mathbb{A}$-algebra homomorphism κ naturally extends to a $\mathbb{Q}(q)$-algebra homomorphism from $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$ to $F_q(\mathfrak{n}(w))$, which we continue to denote by κ. We can now prove our main result:

Theorem 12.3 κ *is an isomorphism from the quantum cluster algebra $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$ to the quantum coordinate ring $A_q(\mathfrak{n}(w))$.*

Proof — By construction, κ is injective. By Proposition 12.1, the image of κ contains $A_q(\mathfrak{n}(w))$. Finally, since κ preserves the Q_+-gradings, and since the homogeneous components of $A_{\mathbb{Q}(q)}(\mathscr{C}_w)$ and $A_q(\mathfrak{n}(w))$ have the same dimensions, by Proposition 12.2, we see that

$$\kappa\left(\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)\right) = A_q(\mathfrak{n}(w)).$$

$\square$

The following Corollary proves the claim made at the end of §7.5.

Corollary 12.4 *All quantum minors $D(b,d)$ $(1 \le b < d \le r)$ belong to $A_q(\mathfrak{n}(w))$, and therefore are polynomials in the dual PBW generators $E^*(\beta_k) = D(k^-,k) = Y_{M_k}$.*

Proof — As shown in the proof of Proposition 12.1, $D(b,d) = \kappa(Y_{M[d,b^+]})$, and so belongs to $\kappa\left(\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)\right) = A_q(\mathfrak{n}(w))$, by Theorem 12.3. □

The following Corollary is a q-analogue of [**GLS6**, Theorem 3.2 (i) (ii)].

Corollary 12.5 *(a) $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$ is an iterated skew polynomial ring.*

(b) The set $\{Y_M(\mathbf{a}) := Y_{M_1}^{a_1} \cdots Y_{M_r}^{a_r} \mid \mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r\}$ is a $\mathbb{Q}(q)$-linear basis of $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$.

Proof — This follows via the isomorphism κ from known properties of $A_q(\mathfrak{n}(w))$. □

Corollary 12.6 *Let $\mathfrak{g}$ be a simple Lie algebra of type A_n, D_n, E_n, and let $\mathfrak{n}$ be a maximal nilpotent subalgebra of $\mathfrak{g}$. Let Λ be the corresponding preprojective algebra of type A_n, D_n, E_n. Then*

(a) The quantum cluster algebra $\mathscr{A}_{\mathbb{Q}(q)}(\mathrm{mod}\,\Lambda)$ is isomorphic to the quantum enveloping algebra $U_q(\mathfrak{n})$.

(b) In this isomorphism, the Chevalley generators e_i $(1 \le i \le n)$ are identified with the quantum cluster variables Y_{S_i} attached to the simple Λ-modules S_i.

Proof — (a) Take w to be the longest element w_0 of the Weyl group. Hence $\mathfrak{n}(w_0) = \mathfrak{n}$, so

$$A_q(\mathfrak{n}(w_0)) = A_q(\mathfrak{n}) \cong U_q(\mathfrak{n}),$$

by Proposition 4.1. On the other hand $\mathscr{C}_{w_0} = \mathrm{mod}(\Lambda)$, and this proves (a). Property (b) is then obvious. □

We believe that Theorem 12.3 can be strengthened as follows.

Conjecture 12.7 *The map κ restricts to an isomorphism from the integral form $\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)$ of the quantum cluster algebra to the integral form $A_{\mathbb{A}}(\mathfrak{n}(w))$ of the quantum coordinate ring.*

12.2 Example

We illustrate our arguments on a simple example. We take $\mathfrak{g}$ of type A_3 and $w = w_0$. We choose the reduced word $\mathbf{i} = (i_6, i_5, i_4, i_3, i_2, i_1) = (1,2,3,1,2,1)$. Therefore

$$\beta_1 = \alpha_1, \ \ \beta_2 = \alpha_1 + \alpha_2, \ \ \beta_3 = \alpha_2, \ \ \beta_4 = \alpha_1 + \alpha_2 + \alpha_3, \ \ \beta_5 = \alpha_2 + \alpha_3, \ \ \beta_6 = \alpha_3.$$

Using the convention of [**GLS6**, §2.4] for visualizing Λ-modules, we can represent the summands of $V_{\mathbf{i}}$ as follows:

$$V_1 = {}_1 \qquad\qquad V_2 = {}^1{}_2 \qquad\qquad V_3 = {}_1{}^2$$

$$V_4 = {}^1{}_2{}_3 \qquad\qquad V_5 = {}_1{}^2{}_2{}_3 \qquad\qquad V_6 = {}_1{}_2{}^3$$

and the summands of $M_{\mathbf{i}}$ as follows:

$$M_1 = {}_1 \qquad\qquad M_2 = {}^1{}_2 \qquad\qquad M_3 = {}_2$$

$$M_4 = {}^1{}_2{}_3 \qquad\qquad M_5 = {}^2{}_3 \qquad\qquad M_6 = {}_3$$

The sequence of mutations of [**GLS6**, §13] consists here in 4 mutations.

Mutation 1: One mutates at V_1 in the maximal rigid module $V_{\mathbf{i}}$. One has

$$\mu_1(V_1) = \mu_1(M[3^-,1]) = M[3,1^+] = M[3,3] = M_3.$$

This gives rise to the two short exact sequences (which can be read from the graph $\Gamma_{\mathbf{i}}$):

$$0 \to V_1 \to V_3 \to M_3 \to 0, \qquad 0 \to M_3 \to V_2 \to V_1 \to 0.$$

By Proposition 10.5, we thus have

$$Y_{M_3}Y_{V_1} = q^{[M_3,V_1]}\left(q^{-1}Y_{V_3} + Y_{V_2}\right) = q^{-1}Y_{V_3} + Y_{V_2},$$

since $[M_3, V_1] = 0$. Using the notation $M[l,k]$, this can be rewritten

$$Y_{M[3,3]}Y_{M[1,1]} = q^{-1}Y_{M[3,1]} + Y_{M[2,2]}. \tag{12.5}$$

Mutation 2: One mutates at V_3 in the maximal rigid module $\mu_1(V_{\mathbf{i}})$. One has

$$\mu_3(V_3) = \mu_3(M[6^-,1]) = M[6,1^+] = M[6,3] = {}_2{}^3.$$

This gives rise to the two short exact sequences (which can be read from the graph $\mu_1(\Gamma_{\mathbf{i}})$):

$$0 \to V_3 \to M_3 \oplus V_6 \to M[6,3] \to 0, \qquad 0 \to M[6,3] \to V_5 \to V_3 \to 0.$$

By Proposition 10.5, we thus have

$$Y_{M[6,3]}Y_{V_3} = q^{[M[6,3],V_3]}\left(q^{-1}Y_{M_3 \oplus V_6} + Y_{V_5}\right) = q^{-1}Y_{M_3}Y_{V_6} + Y_{V_5},$$

since $[M[6,3], V_3] = 0$, and $[M_3, V_6] = 0$. Using the notation $M[l,k]$, this can be rewritten

$$Y_{M[6,3]}Y_{M[3,1]} = q^{-1}Y_{M[3,3]}Y_{M[6,1]} + Y_{M[5,2]}. \tag{12.6}$$

Mutation 3: One mutates at V_2 in the maximal rigid module $\mu_3\mu_1(V_{\mathbf{i}})$. One has

$$\mu_2(V_2) = \mu_2(M[5^-,2]) = M[5,2^+] = M[5,5] = M_5.$$

This gives rise to the two short exact sequences (which can be read from the graph $\mu_3\mu_1(\Gamma_{\mathbf{i}})$):

$$0 \to V_2 \to V_5 \to M_5 \to 0, \qquad 0 \to M_5 \to M_3 \oplus V_4 \to V_2 \to 0.$$

By Proposition 10.5, we thus have

$$Y_{M_5}Y_{V_2} = q^{[M_5,V_2]}\left(q^{-1}Y_{V_5} + Y_{M_3 \oplus V_4}\right) = Y_{V_5} + qY_{M_3}Y_{V_4},$$

since $[M_5, V_2] = 1$, and $[M_3, V_4] = 0$. Using the notation $M[l,k]$, this can be rewritten

$$Y_{M[5,5]}Y_{M[2,2]} = Y_{M[5,2]} + qY_{M[3,3]}Y_{[4,4]}. \tag{12.7}$$

Mutation 4: One mutates at M_3 in the maximal rigid module $\mu_2\mu_3\mu_1(V_i)$. One has

$$\mu_1(M_3) = \mu_1(M[6^-, 3]) = M[6, 3^+] = M[6, 6] = M_6.$$

This gives rise to the two short exact sequences (which can be read from the graph $\mu_2\mu_3\mu_1(\Gamma_i)$):

$$0 \to M_3 \to M[6,3] \to M_6 \to 0, \qquad 0 \to M_6 \to M_5 \to M_3 \to 0.$$

By Proposition 10.5, we thus have

$$Y_{M_6}Y_{M_3} = q^{[M_6, M_3]}\left(q^{-1}Y_{M[6,3]} + Y_{M_5}\right) = q^{-1}Y_{M[6,3]} + Y_{M_5},$$

since $[M_6, M_3] = 0$. Using the notation $M[l,k]$, this can be rewritten

$$Y_{M[6,6]}Y_{M[3,3]} = q^{-1}Y_{M[6,3]} + Y_{M[5,5]}. \tag{12.8}$$

Now, in $A_q(\mathfrak{n})$ we have the following quantum T-system given by Proposition 5.5:

$$D(1,3)D(0,1) = q^{-1}D(0,3) + D(0,2), \tag{12.9}$$

$$D(1,6)D(0,3) = q^{-1}D(1,3)D(0,6) + D(0,5), \tag{12.10}$$

$$D(2,5)D(0,2) = D(0,5) + qD(1,3)D(0,4), \tag{12.11}$$

$$D(3,6)D(1,3) = q^{-1}D(1,6) + D(2,5). \tag{12.12}$$

By definition, the homomorphism $\kappa: \mathscr{A}_{\mathbb{Q}(q)}(\mathrm{mod}\,\Lambda) \to F_q(\mathfrak{n})$ satisfies

$$\kappa(Y_{M[1,1]}) = D(0,1), \quad \kappa(Y_{M[3,1]}) = D(0,3), \quad \kappa(Y_{M[6,1]}) = D(0,6),$$

$$\kappa(Y_{M[2,2]}) = D(0,2), \quad \kappa(Y_{M[5,2]}) = D(0,5), \quad \kappa(Y_{M[4,4]}) = D(0,4).$$

Thus, comparing (12.5) and (12.9) we see that $\kappa(Y_{M[3,3]}) = D(1,3)$. Next, comparing (12.6) and (12.10) we see that $\kappa(Y_{M[6,3]}) = D(1,6)$. Next, comparing (12.7) and (12.11) we see that $\kappa(Y_{M[5,5]}) = D(2,5)$. Finally, comparing (12.8) and (12.12) we see that $\kappa(Y_{M[6,6]}) = D(3,6)$. So in particular, we have

$$\kappa(Y_{M_k}) = D(k^-, k) = E^*(\beta_k), \qquad (1 \le k \le 6).$$

This shows that $\kappa\left(\mathscr{A}_{\mathbb{Q}(q)}(\mathrm{mod}\,\Lambda)\right)$ contains $A_q(\mathfrak{n})$. Comparing dimensions of Q_+-homogeneous components, as in Proposition 12.2, shows that $\kappa\left(\mathscr{A}_{\mathbb{Q}(q)}(\mathrm{mod}\,\Lambda)\right) = A_q(\mathfrak{n}) \cong U_q(\mathfrak{n})$.

In this isomorphism, the Chevalley generators e_1, e_2, e_3 of $U_q(\mathfrak{n})$ correspond to the quantum cluster variables $Y_{M_1}, Y_{M_3}, Y_{M_6}$, respectively. The quantum Serre relation $e_1 e_3 = e_3 e_1$ corresponds to the fact that

$$[M_1, M_6] = [M_6, M_1] = 0, \qquad [M_1, M_6]^1 = 0.$$

To recover, for instance, the relation $e_1^2 e_2 - (q + q^{-1})e_1 e_2 e_1 + e_2 e_1^2 = 0$, we start from the mutation relations

$$Y_{M_3}Y_{M_1} = q^{-1}Y_{V_3} + Y_{V_2}, \qquad Y_{M_1}Y_{M_3} = q^{-1}Y_{V_2} + Y_{V_3}.$$

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The first one is our first mutation above, and the second one is the mutation back, in the opposite direction (mutations are involutive). Eliminating Y_{V_2} between these two equations, we get

$$Y_{M_1}Y_{M_3} - q^{-1}Y_{M_3}Y_{M_1} = (1 - q^{-2})Y_{V_3}.$$

Since $[M_1, V_3]^1 = 0$, $[M_1, V_3] = 1$, and $[V_3, M_1] = 0$, we have

$$Y_{M_1}Y_{V_3} = qY_{V_3}Y_{M_1},$$

which yields

$$Y_{M_1}^2 Y_{M_3} - q^{-1}Y_{M_1}Y_{M_3}Y_{M_1} = qY_{M_1}Y_{M_3}Y_{M_1} - Y_{M_3}Y_{M_1}^2,$$

as expected.

12.3 Canonical basis of $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$

By [**Ki**, §4.7], the subalgebra $U_q(\mathfrak{n}(w))$ is spanned by a subset $\mathbf{B}^*(w)$ of the dual canonical basis $\mathbf{B}^*$ of $U_q(\mathfrak{n})$. Using the isomorphism $\kappa \colon \mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w) \xrightarrow{\sim} A_q(\mathfrak{n}(w)) \cong U_q(\mathfrak{n}(w))$, we can pull back $\mathbf{B}^*(w)$ and obtain a $\mathbb{Q}(q)$-basis of $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$, which we shall call the *canonical basis of* $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$, and denote by $\mathscr{B}(w) = \{b(\mathbf{a}) \mid \mathbf{a} \in \mathbb{N}^r\}$. It may be characterized as follows. Recall the involution σ of $\mathscr{A}_{\mathbb{A}}(\mathscr{C}_w)$ defined in §10.7. We will also denote by σ its extension to $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$.

Proposition 12.8 *For* $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{N}^r$, *the vector* $b(\mathbf{a})$ *is uniquely determined by the following conditions:*

(a) *the expansion of* $b(\mathbf{a})$ *on the basis* $\{Y_M(\mathbf{c}) \mid \mathbf{c} \in \mathbb{N}^r\}$ *is of the form*

$$b(\mathbf{a}) = Y_M(\mathbf{a}) + \sum_{\mathbf{c} \neq \mathbf{a}} \gamma_{\mathbf{a},\mathbf{c}}(q) Y_M(\mathbf{c})$$

where $\gamma_{\mathbf{a},\mathbf{c}}(q) \in q^{-1}\mathbb{Q}[q^{-1}]$ *for every* $\mathbf{c} \neq \mathbf{a}$;

(b) *Let* $\beta(\mathbf{a}) := \sum_{1 \leq k \leq r} a_k \beta_k$. *Then* $\sigma(b(\mathbf{a})) = q^{N(\beta(\mathbf{a}))} b(\mathbf{a})$.

Proof — This is a restatement in our setup of [**Ki**, Theorem 4.26]. The same result was previously obtained in [**La1**] in a particular case. $\qquad\square$

It follows from (10.27) that all quantum cluster monomials satisfy condition (b) of Proposition 12.8. This is similar to [**BZ2**, Proposition 10.9 (2)]. Unfortunately, it is not easy to prove that quantum cluster monomials satisfy (a), so one can only conjecture

Conjecture 12.9 *All quantum cluster monomials* Y_R, *where R runs over the set of reachable rigid modules in* $\mathscr{C}_w$, *belong to* $\mathscr{B}(w)$.

It follows from the original work of Berenstein and Zelevinsky [**BZ1**] that the conjecture holds in the prototypical Example 12.2, namely, for $\mathfrak{g} = \mathfrak{sl}_4$ the dual canonical basis of $U_q(\mathfrak{n})$ is equal to the set of quantum cluster monomials. In this case, there are 14 clusters and 12 cluster variables (including the frozen ones), which all are unipotent quantum minors.

We note that the conjecture is satisfied when $R = M[b,a]$ is one of the modules occuring in the quantum determinantal identities. Indeed, $Y_{M[b,a]}$ is then a quantum minor, and belongs to $\mathscr{B}(w)$ by Proposition 6.3.

It is proved in [**La1**, **La2**] that the conjecture holds for all quantum cluster variables when $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$ is associated with the Kronecker quiver or a Dynkin quiver of type A, and w is the square of a Coxeter element.

Conjecture 12.9 would imply the open orbit conjecture of [**GLS6**, §18.3] for reachable rigid modules, by specializing q to 1. It also appears as Conjecture 1.1 (2) in [**Ki**].

12.4   Quantum coordinate rings of matrices

Let $\mathfrak{g}$ be of type A_n. Let j be a fixed integer between 1 and n, and set $k = n+1-j$. Let w be the Weyl group element with reduced decomposition

$$w = (s_j s_{j-1} \cdots s_1)(s_{j+1} s_j \cdots s_2) \cdots (s_n s_{n-1} \cdots s_k).$$

We denote by $\mathbf{i} = (i_{jk}, i_{jk-1}, \ldots, i_1)$ the corresponding word. It is well known (see [**MC**]) that for this particular choice of $\mathfrak{g}$ and w, $U_q(\mathfrak{n}(w))$ is isomorphic to the quantum coordinate ring $A_q(\mathrm{Mat}(k,j))$ of the space of $k \times j$-matrices.

In this case, $\mathscr{C}_w$ is the subcategory of $\mathrm{mod}\,(\Lambda)$ generated by the indecomposable projective P_k with simple top S_k. For $1 \le a \le k$ and $1 \le b \le j$, the module P_k has a unique quotient X_{ab} with dimension vector $\underline{\dim}\, X_{ab} = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{n+1-b}$. It is not difficult to check that

$$M_{\mathbf{i}} = \bigoplus_{\substack{1 \le a \le k \\ 1 \le b \le j}} X_{ab}.$$

Then, setting $x_{ab} := Y_{X_{ab}} \in \mathscr{A}_q(\mathscr{C}_w)$, we have that the elements x_{ab} satisfy the defining relations of $A_q(\mathrm{Mat}(k,j))$, namely

$$
\begin{aligned}
x_{ab}x_{ac} &= q x_{ac}x_{ab}, & (b < c), && (12.13)\\
x_{ac}x_{bc} &= q x_{bc}x_{ac}, & (a < b), && (12.14)\\
x_{ab}x_{cd} &= x_{cd}x_{ab}, & (a < c,\ b > d), && (12.15)\\
x_{ab}x_{cd} &= x_{cd}x_{ab} + (q-q^{-1})x_{ad}x_{cb}, & (a < c,\ b < d). && (12.16)
\end{aligned}
$$

Thus, Theorem 12.3 gives immediately

Corollary 12.10 *The quantum coordinate ring $A_q(\mathrm{Mat}(k,j))$ is isomorphic to the quantum cluster algebra $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$.*

Example 12.11 The quantum coordinate ring $A_q(\mathrm{Mat}(3,3))$ is isomorphic to $U_q(\mathfrak{n}(w))$ for $\mathfrak{g}$ of type A_5 and $w = s_3 s_2 s_1 s_4 s_3 s_2 s_5 s_4 s_3$. The category $\mathscr{C}_w$ has finite representation type D_4. Taking $\mathbf{i} = (3,2,1,4,3,2,5,4,3)$, the direct indecomposable summands of $V_{\mathbf{i}}$ are

$$V_1 = 3 \qquad\qquad V_2 = \begin{smallmatrix}3\\&4\end{smallmatrix} \qquad\qquad V_3 = \begin{smallmatrix}3\\&4\\&&5\end{smallmatrix}$$

$$V_4 = \begin{smallmatrix}&3\\2\end{smallmatrix} \qquad\qquad V_5 = \begin{smallmatrix}&3\\2&&4\\&3\end{smallmatrix} \qquad\qquad V_6 = \begin{smallmatrix}&3\\2&&4\\&3&&5\\&&4\end{smallmatrix}$$

$$V_7 = \begin{smallmatrix}&&3\\&2\\1\end{smallmatrix} \qquad\qquad V_8 = \begin{smallmatrix}&&3\\&2&&4\\1&&3\\&2\end{smallmatrix} \qquad\qquad V_9 = \begin{smallmatrix}&&3\\&2&&4\\1&&3&&5\\&2&&4\\&&3\end{smallmatrix}$$

The direct indecomposable summands of M_i are

$$M_1 = {}_3 \qquad\qquad M_2 = {}_3{}^{}\,{}_4 \qquad\qquad M_3 = {}^{3}{}_4{}_{\,5}$$

$$M_4 = {}_2{}^{3} \qquad\qquad M_5 = {}_2{}^{3}{}_4 \qquad\qquad M_6 = {}_2{}^{3}{}_4{}_{\,5}$$

$$M_7 = {}_{1}{}_2{}^{3} \qquad\qquad M_8 = {}_{1}{}_2{}^{3}{}_4 \qquad\qquad M_9 = {}_{1}{}_2{}^{3}{}_4{}_{\,5}$$

In the identification of $A_q(\mathrm{Mat}(3,3))$ with $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$, we have

$$x_{11} = Y_{M_9}, \qquad\qquad x_{12} = Y_{M_8}, \qquad\qquad x_{13} = Y_{M_7},$$

$$x_{21} = Y_{M_6}, \qquad\qquad x_{22} = Y_{M_5}, \qquad\qquad x_{23} = Y_{M_4},$$

$$x_{31} = Y_{M_3}, \qquad\qquad x_{32} = Y_{M_2}, \qquad\qquad x_{33} = Y_{M_1}.$$

The quantum cluster algebra $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$ has finite cluster type D_4. Its ring of coefficients is the skew polynomial ring in the variables $Y_{V_3}, Y_{V_6}, Y_{V_7}, Y_{V_8}, Y_{V_9}$. It has 16 non frozen cluster variables, namely the 9 canonical generators Y_{M_i}, the 4 unipotent quantum minors Y_{T_i} attached to the modules:

$$T_1 = {}_{1}{}_2{}^{3}{}_3{}^{4}{}_{5} \qquad T_2 = {}_1{}_2{}^{3}{}_3{}^{4}{}_{5}{}_2 \qquad T_3 = {}_1{}_2{}^{3}{}_3{}^{4}{}_{5}{}_4 \qquad T_4 = {}_1{}_2{}^{3}{}_3{}^{4}{}_{5}{}_2{}_4$$

and the elements Y_{U_i} attached to the 2 modules

$$U_1 = {}_{1}{}_{22}{}^{33}{}_3{}_{44}{}_{5} \qquad\qquad U_2 = {}_1{}_2{}^{3}{}_2{}_{33}{}^{4}{}_4{}_{5}$$

with dimension vector $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$. These cluster variables form 50 clusters.

12.5 Quantum coordinate rings of open cells in partial flag varieties

In this section, we assume that $\mathfrak{g}$ is a simple Lie algebra of simply-laced type. We briefly review some classical material, using the notation of [**GLS5**, **GLS6**].

Let G be a simple simply connected complex algebraic group with Lie algebra $\mathfrak{g}$. Let H be a maximal torus of G, and B, B^- a pair of opposite Borel subgroups containing H with unipotent radicals N, N^-. We denote by $x_i(t)$ $(i \in I, t \in \mathbb{C})$ the simple root subgroups of N, and by $y_i(t)$ the corresponding simple root subgroups of N^-.

We fix a non-empty subset J of I and we denote its complement by $K = I \setminus J$. Let B_K be the standard parabolic subgroup of G generated by B and the one-parameter subgroups

$$y_k(t), \qquad (k \in K, t \in \mathbb{C}).$$

We denote by N_K the unipotent radical of B_K. Similarly, let B_K^- be the parabolic subgroup of G generated by B^- and the one-parameter subgroups

$$x_k(t), \qquad (k \in K, t \in \mathbb{C}).$$

The projective variety $B_K^- \setminus G$ is called a *partial flag variety*. The natural projection map

$$\pi \colon G \to B_K^- \setminus G$$

restricts to an embedding of N_K into $B_K^-\backslash G$ as a dense open subset.

Let W_K be the subgroup of the Weyl group W generated by the reflections s_k ($k \in K$). This is a finite Coxeter group and we denote by w_0^K its longest element. The longest element of W is denoted by w_0. It is easy to check that

$$N_K = N\left(w_0 w_0^K\right), \qquad (12.17)$$

see [**GLS6**, Lemma 17.1]. It follows that $\mathbb{C}[N\left(w_0 w_0^K\right)]$ can be identified with the coordinate ring of the affine open chart $\mathscr{O}_K := \pi(N_K)$ of $B_K^-\backslash G$. Therefore, $A_q(\mathfrak{n}(w_0 w_0^K))$ can be regarded as the quantum coordinate ring $A_q(\mathscr{O}_K)$ of $\mathscr{O}_K$, and Theorem 12.3 implies:

Corollary 12.12 *The quantum coordinate ring $A_q(\mathscr{O}_K)$ is isomorphic to the quantum cluster algebra $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_{w_0 w_0^K})$.*

Example 12.13 Take $G = SL(6)$, and $J = \{3\}$. Then $B_K^-\backslash G$ is the Grassmannian $\mathrm{Gr}(3,6)$ of 3-dimensional subspaces of $\mathbb{C}^6$, and $N_K = N(w_0 w_0^K)$, where $w = w_0 w_0^K$ is as in Example 12.11. Here, $N(w)$ can be identified with the open cell $\mathscr{O}$ of $\mathrm{Gr}(3,6)$ given by the non-vanishing of the first Plücker coordinate. The quantum cluster algebra $\mathscr{A}_{\mathbb{Q}(q)}(\mathscr{C}_w)$ of Example 12.11 can be regarded as the quantum coordinate ring of $\mathscr{O}$.

Example 12.14 Take $G = SO(8)$, of type D_4. We label the vertices of the Dynkin diagram from 1 to 4 so that the central node is 3. Let $J = \{4\}$. Then $B_K^-\backslash G$ is a smooth projective quadric $\mathscr{Q}$ of dimension 6, and N_K can be regarded as an open cell $\mathscr{O}$ in $\mathscr{Q}$. Here $N_K = N(w)$ where $w = w_0 w_0^K = s_4 s_3 s_1 s_2 s_3 s_4$. It is easy to check that the elements Y_{M_k} ($1 \le k \le 6$) satisfy

$$Y_{M_j} Y_{M_i} = q Y_{M_i} Y_{M_j}, \qquad (i < j, \ i+j \ne 7), \qquad (12.18)$$

$$Y_{M_4} Y_{M_3} = Y_{M_3} Y_{M_4}, \qquad (12.19)$$

$$Y_{M_5} Y_{M_2} = Y_{M_2} Y_{M_5} + (q - q^{-1}) Y_{M_3} Y_{M_4}, \qquad (12.20)$$

$$Y_{M_6} Y_{M_1} = Y_{M_1} Y_{M_6} + (q - q^{-1})(Y_{M_2} Y_{M_5} - q^{-1} Y_{M_3} Y_{M_4}), \qquad (12.21)$$

and that this is a presentation of the quantum coordinate ring $A_q(\mathscr{O})$. This shows that $A_q(\mathscr{O})$ is isomorphic to the quantum coordinate ring of the space of 4×4-skew-symmetric matrices, introduced by Strickland [**St**]. The category $\mathscr{C}_w$ has finite representation type $A_1 \times A_1$. Hence, there are 4 cluster variables Y_{M_1}, Y_{M_2}, Y_{M_5}, Y_{M_6}, together with 4 frozen ones, namely, $Y_{V_3} = Y_{M_3}$, $Y_{V_4} = Y_{M_4}$, and

$$Y_{V_5} = Y_{M_2} Y_{M_5} - q^{-1} Y_{M_3} Y_{M_4}, \qquad Y_{V_6} = Y_{M_1} Y_{M_6} - q^{-1} Y_{M_2} Y_{M_5} + q^{-2} Y_{M_3} Y_{M_4}.$$

Observe that Y_{V_6} coincides with the quantum Pfaffian of [**St**]. There are 4 clusters

$$\{Y_{M_1}, \ Y_{M_2}\}, \quad \{Y_{M_1}, \ Y_{M_5}\}, \quad \{Y_{M_6}, \ Y_{M_2}\}, \quad \{Y_{M_6}, \ Y_{M_5}\}.$$

Note that since all quantum cluster variables belong to the basis $\{Y_M(\mathbf{c}) \mid \mathbf{c} \in \mathbb{N}^r\}$, Conjecture 12.9 is easily verified in this case.

Acknowledgments

References

[BFZ]   A. BERENSTEIN, S. FOMIN, A. ZELEVINSKY, *Cluster algebras III: Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), 1–52.

[BZ1]   A. BERENSTEIN, A. ZELEVINSKY, *String bases for quantum groups of type A_r*, in I.M. Gelfand seminar, Adv. Soviet Math. **16**, pp. 51–89, AMS 1993.

[BZ2]   A. BERENSTEIN, A. ZELEVINSKY, *Quantum cluster algebras,* Adv. Math. **195** (2005), 405–455.

[BG]   K. A. BROWN, K. R. GOODEARL, *Lectures on algebraic quantum groups*, Birkhäuser 2002.

[BIRS]   A. BUAN, O. IYAMA, I. REITEN, J. SCOTT, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, Compositio Math. **145** (2009), 1035–1079.

[FZ1]   S. FOMIN, A. ZELEVINSKY, *Double Bruhat cells and total positivity*. J. Amer. Math. Soc. 12 (1999), no. 2, 335–380.

[FZ2]   S. FOMIN, A. ZELEVINSKY, *Cluster algebras. I. Foundations*. J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.

[GLS1]   C. GEISS, B. LECLERC, J. SCHRÖER, *Rigid modules over preprojective algebras*, Invent. Math. **165** (2006), 589–632.

[GLS2]   C. GEISS, B. LECLERC, J. SCHRÖER, *Semicanonical bases and preprojective algebras II: a multiplication formula*, Compositio Math. **143** (2007), 1313–1334.

[GLS3]   C. GEISS, B. LECLERC, J. SCHRÖER, *Auslander algebras and initial seeds for cluster algebras*, J. London Math. Soc. **75** (2007), 718–740.

[GLS4]   C. GEISS, B. LECLERC, J. SCHRÖER, *Cluster algebra structures and semicanonical bases for unipotent groups*, arXiv:math/0703039 (2007), 121p.

[GLS5]   C. GEISS, B. LECLERC, J. SCHRÖER, *Partial flag varieties and preprojective algebras*, Ann. Inst. Fourier (Grenoble) **58** (2008), 825–876.

[GLS6]   C. GEISS, B. LECLERC, J. SCHRÖER, *Kac-Moody groups and cluster algebras,* Adv. Math. **228** (2011), 329–433

[GL]   J. GRABOWSKI, S. LAUNOIS, *Quantum cluster algebra structures on quantum Grassmannians and their quantum Schubert cells: the finite-type cases*, Int. Math. Res. Not., **10** (2011), 2230–2262.

[GLL]   K. GOODEARL, S. LAUNOIS, T. LENAGAN, *Torus-invariant prime ideals in quantum matrices, totally non-negative cells and symplectic leaves*, Math. Z. **269** (2011), 29–45.

[J]   A. JOSEPH, *Quantum groups and their primitive ideals*, Springer 1995. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 29. Springer-Verlag, Berlin, 1995.

[KP]   V. KAC, D. PETERSON, *Regular functions on certain infinite-dimensional groups*, in Arithmetic and Geometry, Vol. II, 141–166, Prog. Math. 36, Birkhäuser 1983.

[K1]   M. KASHIWARA, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63**, (1991), 465–516.

[K2]   M. KASHIWARA, *Global crystal bases of quantum groups*, Duke Math. J. **69**, (1993), 455–485.

[Ki]   Y. KIMURA, *Quantum unipotent subgroup and dual canonical basis*, Kyoto J. Math. **52** (2012), 277–331.

[KNS]   A. KUNIBA, T. NAKANISHI, J. SUZUKI, *T-systems and Y-systems in integrable systems*, J. Phys. A, **44** (2011), 146 p.

[La1] P. LAMPE, *A quantum cluster algebra of Kronecker type and the dual canonical basis*, Int. Math. Res. Not., **13** (2011), 2970–3005.

[La2] P. LAMPE, *Quantum cluster algebras of type A and the dual canonical basis*, arXiv:1101.0580, (2010), 46 p.

[Lu1] G. LUSZTIG, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. 4 (1991), 365–421.

[Lu2] G. LUSZTIG, *Introduction to quantum groups*, Progress in Mathematics, 110. Birkhäuser Boston, Inc., Boston, MA, 1993.

[Lu3] G. LUSZTIG, *Braid group action and canonical bases*, Adv. Math. **122** (1996), 237–261.

[Lu4] G. LUSZTIG, *Semicanonical bases arising from enveloping algebras*, Adv. Math. **151** (2000), 129–139.

[MC] A. MÉRIAUX, G. CAUCHON, *Admissible Diagrams in $U_q^w(g)$ and Combinatoric Properties of Weyl Groups*, Represent. Theory **14** (2010), 645–687.

[Re] M. REINEKE, *Multiplicative properties of dual canonical bases of quantum groups*, J. Algebra **211** (1999), 134–149.

[Ri] C. M. RINGEL, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), 583–591.

[Sa] Y. SAITO, *PBW basis of quantized universal enveloping algebras*, Publ. RIMS, Kyoto Univ. **30** (1994), 209–232.

[St] E. STRICKLAND, *Classical invariant theory for the quantum symplectic groups*, Adv. Math. **123** (1996), 78–90.

[Y] M. YAKIMOV, *Invariant prime ideals in quantizations of nilpotent Lie algebras*, Proc. Lond. Math. Soc. **101** (2010), 454–476.

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