

SEMINAR: BRAUER GROUPS AND THE PERIOD-INDEX PROBLEM, S4A1, SS 2025

The goal of this seminar is to study the Brauer group of general schemes. It consists of three parts:

- (1) Introduction to Brauer groups and Brauer–Severi fibrations;
- (2) Applications of Brauer groups to rationality questions and the Tate conjecture;
- (3) The period-index problem and recent developments.

Organization. This seminar is a combination of a graduate seminar and a research seminar. Credit points can be obtained by following the seminar and either delivering a (joint) talk or submitting an essay on one topic. If you are interested in participating in the seminar, please write an email to *mezzedim@...* before March 14th.

Place and time. Tuesdays 16:00 - 18:00, seminar room 0.011.

1. THE BRAUER GROUP OF A FIELD

April 8th, Veran Stojanovic

References: [4, Chapter 1.1–1.3], [5, Chapters 1–3]. Briefly recall the language of Galois cohomology. Central simple algebras, Wedderburn theorem. Definition of the Brauer group via central simple algebras and cohomological description of $\mathrm{Br}(k)$ in terms of Galois cohomology. Kummer sequence.

2. THE BRAUER GROUP OF A SCHEME

April 15th, Javier Fernández Píriz & Edward Young

References: [7, 8, 9], [4, Chapter 3]. Azumaya algebras and the Brauer–Azumaya group. Briefly recall the language of étale cohomology. The cohomological Brauer group via étale cohomology and comparison of the two Brauer groups (only the statement, we will see the proof in Talk 6). Kummer sequence. Unramified Brauer classes. Brauer group of smooth curves. Purity for the Brauer group.

3. BRAUER–SEVERI VARIETIES I

April 22nd, Qinxiang Hou & Wenrong Zou

References: [2], [4, Section 6.1], [5, Chapter 5]. Brauer–Severi varieties over a field k as twists of \mathbb{P}_k^n . A Brauer–Severi variety is trivial if and only if it has a k -point. Brauer–Severi varieties over general schemes, and conic/quadric

bundles. Period and index of a Brauer–Severi variety, and relationship between period and index.

4. GERBES AND TWISTED SHEAVES

April 29th, Maxim Brais & Lennart Christian Grabbel

References: [17, Sections 12.2, 12.3], [6], [4, Section 2.6.4 and 4.2]. Give a self-contained account of gerbes and twisted sheaves, with particular emphasis on the geometric side. Use it to explain de Jong’s proof of Gabber’s theorem comparing the cohomological Brauer group and the Brauer–Azumaya group.

5. UNI(RATIONALITY) AND THE ARTIN–MUMFORD EXAMPLE

May 6th, Omer Bojan

References: [4, Chapters 5, 10, 11]. The Brauer group of a smooth projective variety is a birational invariant. The Artin–Mumford example: a 3-fold that is unirational, but not stably rational. Rationality in families by Hassett–Pirutka–Tschinkel.

6. BRAUER–SEVERI VARIETIES II

May 13th, Hannah Dell & Brais Gerpe Vilas & Gebhard Martin

References: [3]. Explain the (integral and rational) cohomology of a Brauer–Severi variety over a scheme X (Note: These results have not been written up in the literature, so the person in charge would have to dig deep in the literature and probably prove some results themselves. The goal would be to have a description of the cohomology via a twisted version of Leray–Hirsch. For Brauer–Severi varieties over a field, much more is written down). Explain Bernardara’s result on the derived category of coherent sheaves on a Brauer–Severi variety.

7. BRAUER GROUPS AND THE TATE CONJECTURE

May 20th, Moritz Hartlieb

References: [4, Chapter 16], [20], [19], [18]. State the Tate conjecture, and explain how it is related to the finiteness of the Brauer group for smooth projective varieties over finite fields. Prove that, for a K3 surface X over a number field k , the group $\mathrm{Br}(\overline{X})^{\mathrm{Gal}(\overline{k}/k)}$ is finite (and its cardinality can be explicitly bounded). If time permits, talk about the Artin–Tate pairing on the l -primary part of the quotient $\mathrm{Br}(X)/\mathrm{Br}(X)_{\mathrm{div}}$, where X is a smooth projective surface over a finite field.

8. PERIOD-INDEX PROBLEM IN DIMENSION 2: CHARACTERISTIC 0

May 27th, Dario Faro & Anna Savelyeva

References: [14], [21]. Introduce the period-index problem, and explain why it is obvious for curves. Explain de Jong's proof of the period-index theorem for surfaces in characteristic 0.

9. PERIOD-INDEX PROBLEM IN DIMENSION 2: CHARACTERISTIC p *June 3rd, Giacomo Mezzedimi*

References: [16]. Explain Lieblich's proof of the period-index theorem for fields of transcendence degree 2 over \mathbb{F}_p .

10. PERIOD-INDEX PROBLEM FOR UNRAMIFIED CLASSES

June 17th, Frank Gounelas & Daniel Huybrechts

References: [15]. Explain de Jong–Starr's result and its important consequence: if the period-index problem holds for *unramified* classes of an arbitrary variety, then it holds for all Brauer classes.

11. SPLITTING BRAUER CLASSES BY ABELIAN VARIETIES

June 24th, Riccardo Carini

References: [10, 1, 12, 13] Here we can choose a subset of the referenced works. The first two papers show how to split Brauer classes of degree ≤ 7 via genus one curves, while the second two papers prove the splitting of (unramified) Brauer classes via (torsors of) abelian varieties.

12. THE PERIOD-INDEX PROBLEM FOR ABELIAN THREEFOLDS

July 8th, Alessio Bottini

References: [11]. Explain the work of Hotchkiss–Perry, proving the period-index conjecture for unramified classes on abelian threefolds.

REFERENCES

- [1] B. Antieau, A. Auel, *Explicit descent on elliptic curves and splitting Brauer classes*. Preprint, arXiv:2106.04291 [math.NT] (2021).
- [2] M. Artin, *Brauer-Severi varieties*. Brauer groups in ring theory and algebraic geometry, Proc., Antwerp 1981, Lect. Notes Math. 917, 194-210 (1982).
- [3] M. Bernardara, *A semiorthogonal decomposition for Brauer-Severi schemes*. Math. Nachr. 282, No. 10, 1406-1413 (2009).
- [4] J.-L. Colliot-Thélène, A. Skorobogatov, *The Brauer–Grothendieck group*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge 71. xv, 453 p. (2021).
- [5] P. Gille, T. Szamuely, *Central simple algebras and Galois cohomology*. Cambridge Studies in Advanced Mathematics 165. Cambridge: Cambridge University Press. xi, 417 p. (2017).
- [6] J. Gireaud, *Cohomologie non abélienne*. Die Grundlehren der mathematischen Wissenschaften. Band 179. Berlin-Heidelberg-New York: Springer-Verlag. ix, 467 p. (1971).

- [7] A. Grothendieck, *Le groupe de Brauer. I: Algèbres d'Azumaya et interprétations diverses*. Sémin. Bourbaki Vol. 9, 17e année (1964/1965), Exp. No. 290, 21 p. (1966).
- [8] A. Grothendieck, *Le groupe de Brauer. II: Théorie cohomologique*. Sémin. Bourbaki 1965/66, Exp. No. 297, 21 p. (1966).
- [9] A. Grothendieck, *Le groupe de Brauer. III: Exemples et compléments*. Dix Exposés Cohomologie Schémas, Adv. Stud. Pure Math. 3, 88-188 (1968).
- [10] W. Ho, A. J. de Jong, *Genus one curves and Brauer-Severi varieties*. Math. Res. Lett. 19, No. 6, 1357-1359 (2012).
- [11] J. Hotchkiss, A. Perry, *The period-index conjecture for abelian threefolds and Donaldson-Thomas theory*. Preprint, arXiv:2405.03315 [math.AG] (2024).
- [12] W. Ho, M. Lieblich, *Splitting Brauer classes using the universal Albanese*. Enseign. Math. (2) 67, No. 1-2, 209-224 (2021).
- [13] D. Huybrechts, D. Mattei, *Splitting unramified Brauer classes by abelian torsors and the period-index problem*. Preprint, arXiv:2310.04029 [math.AG] (2023).
- [14] A. J. de Jong, *The period-index problem for the Brauer group of an algebraic surface*. Duke Math. J. 123, No. 1, 71-94 (2004).
- [15] A. J. de Jong, J. Starr, *Almost proper GIT-stacks and discriminant avoidance*. Doc. Math. 15, 957-972 (2010).
- [16] M. Lieblich, *The period-index problem for fields of transcendence degree 2*. Ann. Math. (2) 182, No. 2, 391-427 (2015).
- [17] M. Olsson, *Algebraic spaces and stacks*. Colloquium Publications. American Mathematical Society 62. Providence, RI: American Mathematical Society (AMS). xi, 298 p. (2016).
- [18] M. Orr, A. Skorobogatov, *Finiteness theorems for K3 surfaces and abelian varieties of CM type*. Compos. Math. 154, no. 8, 1571-1592 (2018).
- [19] A. Skorobogatov, Y. Zarhin, *A finiteness theorem for the Brauer group of abelian varieties and K3 surfaces*. J. Algebraic Geom. 17, no. 3, 481-502 (2008).
- [20] J. Tate, *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog*. Séminaire Bourbaki, Vol. 9, Exp. No. 306, 415-440 Société Mathématique de France, Paris (1995).
- [21] M. Van den Bergh, *Notes on De Jong's period=index theorem for central simple algebras over fields of transcendence degree two*. Preprint, arXiv:0807.1403 [math.AG] (2008).